# Optimal Point Estimates for Multi-Target States based on Kernel Distances

**Marcus Baum**, **Patrick Ruoff**, **Dominik Itte**, and **Uwe D. Hanebeck**
Intelligent Sensor-Actuator-Systems Laboratory (ISAS)
Institute for Anthropomatics
Karlsruhe Institute of Technology (KIT), Germany
marcus.baum@kit.edu, ruoff@kit.edu, dominik.itte@student.kit.edu, uwe.hanebeck@ieee.org

***Abstract*— Almost all multi-target tracking systems have to generate point estimates for the targets, e.g., for displaying the tracks. The novel idea in this paper is to consider point estimates for multi-target states that are optimal according to a kernel distance measure. Because the kernel distance is a metric on point sets and ignores the target labels, shortcomings of Minimum Mean Squared Error (MMSE) estimates for multi-target states can be avoided. We show how the calculation of these point estimates can be casted as an optimization problem and it turns out that it corresponds to the problem of reducing the Probability Hypothesis Density (PHD) function to a Dirac mixture density. Finally, we discuss a generalization of the kernel distance called LCD distance, which does not require to choose a specific kernel width. The presented methods are evaluated in a Multiple-Hypotheses Tracker (MHT) setting with up to ten targets.**

## I. Introduction

Multi-target tracking [1] deals with the problem of estimating the kinematic parameters of multiple targets based on noisy measurements while the measurement-to-target association is unknown. A multi-target tracking system usually has to generate point estimates for the target states [2], e.g., for displaying an operator or decision making. Unfortunately, the widely-used global Minimum Mean Squared Error (MMSE) estimate turned out to be unsuitable for multi-target tracking needs as demonstrated in the following example.

**Example 1.** Consider two one-dimensional targets with state variables $\underline{x}_{k,1}$ and $\underline{x}_{k,2}$, where $k$ denotes the time index. A multi-target tracking algorithm such as the *Multiple Hypotheses Tracker (MHT)* [3] may maintain a probability density function for the stacked vector $[x_{k,1}, x_{k,2}]^T$, i.e.,

$$p(x_{k,1}, x_{k,2}|\mathcal{Y}_k) \ ,$$

where $\mathcal{Y}_k$ denotes all available measurements up to time step $k$. Fig. 1 shows an example for the probability density function, which in this case consists of a Gaussian mixture with two components. This is a typical situation, which arises when two targets are close to each other, because then the target identities get lost. Unfortunately, the mean of the density, i.e., the MMSE estimate, is given by $[1.5, 1.5]^T$, which says both targets are at position 1.5. This so-called *Mixed Labelling Problem* [4], [5] is a serious problem as the positions of the targets are actually known more precisely, only their identities/labels are not known.

In this work, we consider methods for determining point estimates when we are not interested in the target labels; we do only want to know their positions. In this manner, the coalescence of the trackes can be prevented. Note that there is a variety of applications in which the target identities themselves are not relevant, e.g. collision avoidance and group target tracking.

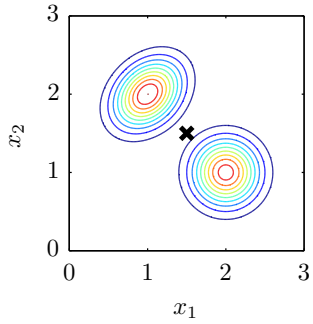



Fig. 1: Example of a joint density for two target states $x_1$ and $x_2$. The mean of the density is $[1.5, 1.5]^T$.

## II. Problem Formulation

We consider the problem of determining point estimates for multi-target states that are represented with a probability density for the stacked single target states or with a Probability Hypothesis Density (PHD) function. We assume the number of targets to be given, e.g., an estimate of the target number is available. Furthermore, we will omit the time index $k$ in the following.

*Stacked Single Target States:* A wide class of Bayesian multi-target tracking algorithms represent the state as an ordered random vector. $\underline{\boldsymbol{\chi}} = \left[\underline{\boldsymbol{x}}_1^T, \ldots, \underline{\boldsymbol{x}}_N^T\right]^T$, where $\underline{\boldsymbol{x}}_i \in \mathbb{R}^n$ denotes the state vector of target $i$ (with $1 \leq i \leq N$). The tracking algorithm maintains a probability density function for $\underline{\boldsymbol{x}}$ conditioned on the received data $\mathcal{Y}$

$$p\left(\underline{\chi}|\mathcal{Y}\right) \ . \tag{1}$$

For instance, a *Multiple Hypotheses Tracker (MHT)* [3] maintains Gaussian mixture, and particle filters methods [6] are based on a particle approximation. A point estimate is supposed to be optimal in the sense that it minimizes the expectation of risk function, e.g., well-known Minimum Mean-Squared-Error (MMSE) estimate [1] is defined as $\hat{\underline{\chi}}^{\text{MMSE}} := \arg\min_{\hat{\underline{\chi}}} \mathrm{E}\{||\hat{\underline{\chi}} - \underline{\boldsymbol{\chi}}||^2|\mathcal{Y}\} = \mathrm{E}\{\underline{\boldsymbol{\chi}}|\mathcal{Y}\}$.

*Probability Hypothesis Density (PHD):* Besides multi-target tracking algorithms that estimate the stacked single target states, there are algorithms such as the *Probability Hypotheses Density (PHD)* filter [7] that manage a Probability Hypothesis Density (PHD) $D(\underline{x})$, which is the first-order

posterior multi-target moment. For a given number of targets, the PHD can be computed from (1) as follows

$$D(\underline{x}|\mathcal{Y}) \quad = \quad \sum_i p(\underline{x}_i|\mathcal{Y}) \ . \tag{2}$$

## III. Related Work

In [8], [9], [10], [11] it was suggested to employ *Minimum Mean OSPA* estimates, where *OSPA* denotes the *Optimal Subpattern Assignment Metric* [12]. In [4], [5], *Maximum A Posteriori (MAP)* estimates were used instead of the MMSE estimate. In [13] it was shown that each symmetric state transformations on multiple target states defines a metric on the set of unlabeled target states. For those metrics, optimal estimates can be computed rather easily by performing probabilistic forward inference. For a PHD that is represented with particles, a point estimate is typically extracted with a clustering algorithm such as k-means or expectation maximization [14], [15]. For PHDs represented with Gaussian mixtures, the means of the Gaussians with the largest weights can serve as point estimates [7].

## IV. Kernel Distance for Multi-Target States

As pointed out in Section I, MMSE estimates may be unsuitable for multi-target states as the MMSE incorporates the labels. Hence, we are looking for alternative risk functions that do not take the labeling of the targets into account. More precisely, for two multi-target states $\underline{\chi} = \left[\underline{x}_1^T, \ldots, \underline{x}_N^T\right]^T$ and $\underline{\nu} = [\underline{y}_1^T, \ldots, \underline{y}_N^T]^T$, we need a distance function $d(\cdot,\cdot)$, which satisfies

$$d(\underline{\chi}, \underline{\nu}) = d(\underline{\chi}, \underline{\nu}^\pi) \ , \text{ for all } \pi \in \Pi^N \tag{3}$$

where $\underline{\nu}^\pi$ denotes the vector $\underline{\nu}$ with permuted single target states according to the permutation $\pi$, i.e., $\underline{\nu}^\pi := [\underline{y}_{\pi(1)}^T, \ldots, \underline{y}_{\pi(N)}^T]^T$.

First, we will employ the kernel distance for constructing such a risk function. As the kernel distance requires to choose a specific kernel width, we will later in Section VIII consider a generalization called *LCD distance*.

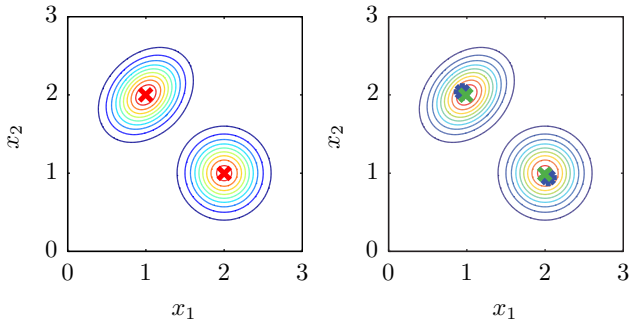
The kernel distance is a widely-used distance measure [16], [17]. For example, in machine learning probability distributions [16] are compared with the kernel distance and in shape analysis [18] the kernel distance is used for point sets, curves or surfaces. One of the main benefits of the kernel distance, besides of its theoretical justification, are its intuitive meaning and low computational complexity. In order to introduce the kernel distance for multi-target states, we start with a more general definition of the kernel distance for distribution functions.

**Definition 1** (Kernel Distance)**.** The kernel distance between two distributions $f_1(\underline{x})$ and $f_2(\underline{x})$ with $\underline{x} \in \mathbb{R}^n$ is defined as

$$d_{\text{Kernel}}(f_1, f_2)^2 := ||F_1 - F_2||_{L^2}^2 = \int_{\mathbb{R}^n} \left(F_1(\underline{m}) - F_2(\underline{m})\right)^2 \mathrm{d}\underline{m} \ , \tag{4}$$

where the so-called *kernel density* $F_i$ for $i \in \{1,2\}$ denotes the density $f_i$ after the convolution with a kernel function $K^b(\cdot,\cdot)$, i.e.,

$$F_i(\underline{m}) := \int_{\mathbb{R}^n} f(\underline{x})\, K^b(\underline{x} - \underline{m})\, \mathrm{d}\underline{x} \ . \tag{5}$$




(a) MMOSPA estimate (exponent 2 and no cut-off value).

(b) MMKD estimate for kernel width 1 (green cross) and 0.5 (blue asterisk).

Fig. 2: Comparison of MMOSPA and MMKD estimate.

*Remark* 1. In this work, we only consider an isotropic (unnormalized) Gaussian kernel function

$$K^b(\underline{x} - \underline{m}) = \exp\left(-\frac{1}{2}\frac{(\underline{x} - \underline{m})^T(\underline{x} - \underline{m})}{b^2}\right)$$

with one-dimensional parameter $b$ for the width.

Based on the above general definition of the kernel distance for distributions, we can define the kernel distance for multi-target states, i.e., a set of single target states, by interpreting the multi-target state as a Dirac mixture distribution.

**Definition 2** (Kernel Distance for Multi-Target States)**.** The kernel distance between two multi-target states $\underline{\chi} = \left[\underline{x}_1^T, \ldots, \underline{x}_N^T\right]^T$ and $\underline{\nu} = [\underline{y}_1^T, \ldots, \underline{y}_n^T]^T$ is defined as the kernel distance between the two Dirac distributions $f_{\underline{\chi}}(\underline{x}) := \sum_i \delta(\underline{x} - \underline{x}_i)$ and $f_{\underline{\nu}}(\underline{x}) := \sum_i \delta(\underline{x} - \underline{y}_i)$, i.e.,

$$d_{\text{Kernel}}(\underline{\chi}, \underline{\nu})^2 := ||F_{\underline{\chi}} - F_{\underline{\nu}}||_{L^2}^2 \ , \tag{6}$$

where $F_{\underline{\chi}}$ and $F_{\underline{\nu}}$ denote the *kernel densities* of $f_{\underline{\chi}}(\underline{x})$ and $f_{\underline{\nu}}(\underline{x})$ after a convolution with a kernel function.

*Remark* 2. The kernel distance between the two multi-target state in Definition 2 can be written as

$$d_{\text{Kernel}}(\underline{\chi}, \underline{\nu})^2 = \text{const} \cdot \left( \sum_i \sum_j K^{\sqrt{2}b}(\underline{x}_i - \underline{x}_j) - 2\sum_i \sum_j K^{\sqrt{2}b}(\underline{x}_i - \underline{y}_j) + \sum_i \sum_j K^{\sqrt{2}b}(\underline{y}_i - \underline{y}_j) \right) \ , \tag{7}$$

*Remark* 3. It is obvious that the kernel distance satisfies the permutation invariance defined in (3).

The kernel distance has an intuitive meaning for multi-target states. Essentially, the kernel is placed at each single target state and then the $L^2$ distance of the resulting functions is computed.

## V. Minimum Mean Kernel Distance (MMKD) Estimate

Based on the kernel distance, we can define the *Mean Kernel Distance* to a point estimate.

*Remark* 4. For the sake of simplicity, we do not explicitly state that the densities are conditioned on the data $\mathcal{Y}$, i.e., we use $p\left(\underline{\chi}\right)$ for $p\left(\underline{\chi}|\mathcal{Y}\right)$.

**Definition 3** (MKD)**.** The *Mean Kernel Distance (MKD)* to a point estimate $\hat{\underline{\chi}}$ and the density $p\left(\underline{\chi}\right)$ is defined as

$$\mathrm{MKD}(\hat{\underline{\chi}}) := \mathrm{E}\{d^2_{\text{Kernel}}(\hat{\underline{\chi}}, \underline{\boldsymbol{\chi}})\} \quad . \tag{8}$$

**Definition 4** (MMKD)**.** The *Minimum Mean Kernel Distance (MMKD)* estimate for $p\left(\underline{\chi}\right)$ is given by

$$\hat{\underline{\chi}}^{\mathrm{MMKD}} := \arg\min_{\hat{\underline{\chi}}} \mathrm{E}\{d^2_{\text{Kernel}}(\hat{\underline{\chi}}, \underline{\boldsymbol{\chi}})\} \quad .$$

*Remark* 5. All permutations of $\hat{\underline{\chi}}^{\mathrm{MMKD}}$ are also MMKD estimates (because the order does not matter). Hence, there are $N!$ equal MMKD estimates.

Note that these definitions are analogous to the *Mean OSPA (MOSPA)* [8] and *Minimum Mean OSPA (MMOSPA)* estimate [8]. For well-separated targets, the MMKD estimate coincides with the MMSE estimate. We argue that the kernel width has an influence on the resulting point estimate. In case $b \to \infty$, each $\hat{\underline{\chi}} \in \mathbb{R}^{N\times n}$ is a minimum, and when $b \to 0$, the single target estimates will be located at the maximum of the PHD (see the next section).

*Discussion:* A main benefit of the kernel distance is that it can be computed by means of adding kernels. As a consequence, the mean kernel distance to a point estimate (8) and its derivative can be calculated in closed form. We will show that it is no problem to extract point estimates for ten targets with respect to the kernel distance in Section IX. Experiments indicate that MMKD estimates can serve as good approximation for MMOSPA estimates. The MMKD estimate can be extracted from the corresponding PHD (see the next section). Hence, the MMKD estimate offers a theoretical well-grounded way for extracting point estimates from a PHD.

**Example 2.** Fig. 2 compares the MMOSPA estimate with the MMKD estimate for kernel widths $0.5$ and $1$. The MMOSPA estimate [8], [9], [10] was computed using an exponent of 2 and no cut-off value. Note again that there are two MMOSPA and MMKD estimates as the target states can be switched. It can be seen that the MMKD estimate is close to the MMOSPA estimate for both kernel widths. Furthermore, the influence of the kernel width is not significant.

## VI. MMKD Estimate for a PHD

A main result is that the *Minimum Mean Kernel Distance (MMKD)* estimate for posterior probability density $p(\underline{\chi})$ can be computed by considering the corresponding PHD. In order to prove this, we first show that the expected kernel density equals the kernel density of the PHD.

**Theorem 1.** *For a multi-target state vector* $\underline{\boldsymbol{\chi}} = \left[\underline{\boldsymbol{x}}_1^T, \ldots, \underline{\boldsymbol{x}}_N^T\right]^T$*, where* $\underline{\boldsymbol{x}}_i \in \mathbb{R}^n$ *with probability density* $p(\underline{\chi})$ *and corresponding (uncertain ) Dirac density* $f_{\underline{\boldsymbol{\chi}}}(\underline{x}) = \sum_i \delta(\underline{x} - \underline{\boldsymbol{x}}_i)$*, the following holds*

$$\mathrm{E}\{F_{\underline{\boldsymbol{\chi}}}(\cdot)\} = F_{D_{\underline{\boldsymbol{\chi}}}}(\cdot) \ .$$

Proof.

$$\begin{aligned}
\mathrm{E}\{F_{\underline{\boldsymbol{x}}}(\underline{s})\} &= \int F_{\underline{x}}(\underline{s}) p(\underline{x}) d\underline{x} \\
&= \int\int \sum_i \delta(\underline{x}^* - \underline{x}_i) \cdot K^b(\underline{x}^* - \underline{s}) \, d\underline{x}^* \, p(\underline{x}) d\underline{x} \\
&= \int\int \sum_i \delta(\underline{x}^* - \underline{x}_i) \, p(\underline{x}) d\underline{x} K^b(\underline{x}^* - \underline{s}) \, d\underline{x}^* \\
&= \int D_{\underline{\boldsymbol{\chi}}}(\underline{s}) K^b(\underline{x}^* - \underline{s}) \, d\underline{x}^* \ = \ F_{D_{\underline{\boldsymbol{\chi}}}}(\underline{s})
\end{aligned}$$

□

Based on the above theorem, we can show that extracting the point estimate corresponds to finding the optimal approximation of the PHD with $N$ Diracs according to the kernel distance. This highlights an interesting connection between probability density approximation and point estimates.

**Theorem 2.** *For a multi-target state random vector* $\underline{\boldsymbol{\chi}} = \left[\underline{\boldsymbol{x}}_1^T, \ldots, \underline{\boldsymbol{x}}_N^T\right]^T$*, where* $\underline{\boldsymbol{x}}_i \in \mathbb{R}^n$ *with probability density* $p(\underline{x})$

$$\hat{\underline{\chi}}^{\mathrm{MMKD}} = \arg\min_{\hat{\underline{\chi}}} d_{\mathrm{Kernel}}(f_{\hat{\underline{\chi}}}, D_{\underline{\boldsymbol{\chi}}}) \ . \tag{9}$$

Proof. Per definition for all $\hat{\underline{\chi}}$, we have

$$\begin{aligned}
&\mathrm{E}\{d_{\text{Kernel}}(\hat{\underline{\chi}}, \underline{\boldsymbol{\chi}})\} \\
&= \int\int \left(F_{\hat{\underline{\chi}}}(\underline{m}) - F_{\underline{\boldsymbol{\chi}}}(\underline{m})\right)^2 \mathrm{d}\underline{m} \cdot p(\underline{\chi}) \, \mathrm{d}\underline{\chi} \\
&\quad \int\int \left(F_{\hat{\underline{\chi}}}(\underline{m})\right)^2 - 2 \cdot F_{\underline{\boldsymbol{\chi}}}(\underline{m}) \cdot F_{\hat{\underline{\chi}}}(\underline{m}) \\
&\quad + F_{\underline{\boldsymbol{\chi}}}(\underline{m})^2 \, \mathrm{d}\underline{m} \cdot p(\underline{\chi}) \, \mathrm{d}\underline{\chi} \ .
\end{aligned}$$

The outer integral can be switched with the inner one and terms independent of $\hat{\underline{\chi}}$ can be summarized in a factor $c_1$

$$\int F_{\hat{\underline{\chi}}}(\underline{m})^2 - 2 \cdot \mathrm{E}\{F_{\underline{\boldsymbol{\chi}}}(\underline{m})\} \cdot F_{\hat{\underline{\chi}}}(\underline{m}) \mathrm{d}\underline{m} + c_1$$

A factorization then yields the desired term

$$\begin{aligned}
&\int \left(F_{\hat{\underline{\chi}}}(\underline{m}) - \mathrm{E}\{F_{\underline{\boldsymbol{\chi}}}(\underline{m})\}\right)^2 + c - \mathrm{E}\{F_{\underline{\boldsymbol{\chi}}}(\underline{m})\} \, \mathrm{d}\underline{m} \\
&= \ d_{\text{Kernel}}(f_{\hat{\underline{\chi}}}, D_{\underline{\boldsymbol{\chi}}}) + c_2 \ ,
\end{aligned}$$

where $c_2$ is a constant independent of $\hat{\underline{\chi}}$. □

## VII. Computing the MMKD Estimate

Based on Theorem 2, we can define an optimization problem for finding the MMKD estimate. If a probability density for stacked single target states (1) is given, first the corresponding PHD has to be computed according to (2). Then the MMKD estimate can be found by solving the minimization problem specified in Equation (9). It is important to note that there are closed-form expressions for evaluating the kernel distance from $f_{\hat{\underline{\chi}}}$ to the PHD $D_{\underline{\boldsymbol{\chi}}}$ given by $d_{\text{Kernel}}(f_{\hat{\underline{\chi}}}, F_{D_{\underline{\boldsymbol{\chi}}}})$ in case the PHD is given by a Gaussian mixture. The reason is that, if the PHD is Gaussian mixture $F_{D_{\underline{\boldsymbol{\chi}}}}$ is a Gaussian mixture as well and explicit formulas for computing the kernel distance between two Gaussian mixtures are for example given in [18], [19]. It is even possible to

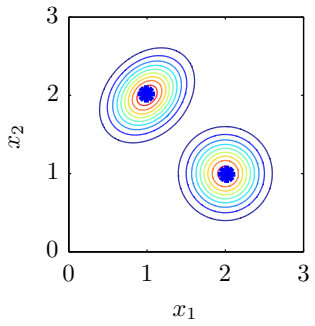



Fig. 3: MMLCD estimate for Example 1.

calculate the derivative of $d_{\text{Kernel}}(f_{\hat{\underline{\chi}}}, D_{\underline{\boldsymbol{\chi}}})$ with respect to $\hat{\underline{\chi}}$, which is very useful for many optimization algorithms such as Newton methods [19].

## VIII. A Generalization of the Kernel Distance - The LCD Distance

The kernel distance requires to specify a kernel width. In order to overcome this issue, we propose to use a generalization of the kernel distance, the so-called *Localized Cumulative Distribution (LCD)*, which was introduced [20] for the sake of comparing multivariate probability densities. The basic idea is to use all kernel widths and not only one specific width. Hence, the kernel density (5) now also depends on both the kernel width and kernel center [20], [19].

**Definition 5.** For a density $f(\underline{x})$, the LCD [20] is defined as the function $F(.,.) : \mathbb{R}^n \times \mathbb{R}_+ \to [0,1]$ with

$$F^{\text{LCD}}(\underline{m}, \underline{b}) = \int_{\mathbb{R}^n} f(\underline{x})\, K^b(\underline{x} - \underline{m})\, \mathrm{d}\underline{x} \ , \qquad (10)$$

where $K^b(\underline{x} - \underline{m})$ is a suitable kernel with location $\underline{m}$ and width $\underline{b} \in \mathbb{R}_+$.

Based on the LCD, we can define a distance measure on densities using weighted $L^2$ distance of two LCDs.

**Definition 6** (*Modified Cramér–von Mises Distance* [20])**.** The distance $d_{\text{LCD}}(f_1, f_2)$ between two densities $f_1$ and $f_2$ is defined by means of their LCDs $F_1^{\text{LCD}}(\underline{m}, b)$ and $F_2^{\text{LCD}}(\underline{m}, b)$ as [20]

$$d_{\text{LCD}}(f_1, f_2)^2 := ||F_1^{\text{LCD}} - F_2^{\text{LCD}}||^2_{L^2(w)} = \int_{\mathbb{R}_+} w(b) \int_{\mathbb{R}^n} \left(F_1^{\text{LCD}}(\underline{m}, b) - F_2^{\text{LCD}}(\underline{m}, b)\right)^2 \mathrm{d}\underline{m}\, \mathrm{d}b \ , \qquad (11)$$

where $w(b) : \mathbb{R}_+ \to [0,1]$ is a suitable weighting function.

A suitable weighting function is employed for decreasing the influence of large kernel widths [20]. The LCD distance on sets is defined in the same way as (2) by interpreting the sets as Dirac densities. The LCD has an intuitive interpretation as a distance measure on sets: Two sets are similar if a particular region (specified by the kernel) contains a similar number of points. This is a generalization of a kernel distance [16] as a usual kernel distance is obtained if the weighting function is a point mass. The *Minimum Mean LCD estimate (MMLCD)* is defined in analogy to Definition 2. Note that proper versions of Theorem 2 and Theorem 1 for the LCD distance can also be proven. Hence, the MMLCD estimate can be obtained by approximating the PHD with a Dirac mixture according to the LCD distance. A very intuitive distance measure for multi-target states is obtained if the weighting factor $w(b)$ in (11) is chosen to be $\frac{1}{b^N}$.

**Definition 7** (LCD Distance for multi-target states)**.** The LCD distance between two multi-target states $\underline{\chi} = \left[\underline{x}_1^T, \ldots, \underline{x}_N^T\right]^T$ and $\underline{\nu} = [\underline{y}_1^T, \ldots, \underline{y}_n^T]^T$

$$d_{\text{LCD}}(\underline{\chi}, \underline{\nu})^2 := -\Bigg( \sum_i \sum_j \|\underline{x}_i - \underline{x}_j\| - 2 \sum_i \sum_j \|\underline{x}_i - \underline{y}_j\| + \sum_i \sum_j \|\underline{y}_i - \underline{y}_j\| \Bigg) \ . \qquad (12)$$

The main benefit of this distance measure is that it does not depend on a kernel width and does not contain any minimization over permutations. However, there are no closed form expressions for calculating the LCD distance between a Dirac mixture density and an arbitrary Gaussian mixtures, i.e., for Equation (6). Nevertheless, (6) can be calculated in closed form for Dirac mixture densities [19] and also the derivates are available.

**Example 3.** Fig. 3 shows the MMLCD estimate for Example 1. It can be seen that the MMKD estimate is close to the MMOSPA and MMKD estimates in Fig. 2.

## IX. Evaluation

For the purpose of evaluation, we use a Multiple Hypotheses Tracker (MHT) for tracking multiple targets based on noisy position measurements. The single target motion is modeled with a constant velocity model (variance of the system noise: $\text{diag}([0.076, 0.076])$) and Cartesian position measurements corrupted with additive Gaussian noise with variance $\text{diag}([0.5, 0.5])$ are obtained. The MHT tracker maintains a Gaussian mixture density for the stacked single target states (1). The methods for extracting point estimates are only used for displaying the tracks.

*Scenario 1: Ten Targets:* This scenario demonstrates that the MMKD estimates and MMLCD estimates are suitable for a large number of targets, i.e. ten targets. Fig. 4 shows the tracks of ten targets plus the position measurements received from the targets. The plotted point estimates for the target positions in Fig. 7b are the MMSE estimates. Exactly the same probability densities were used for extracting MMLCD estimates in Fig. 7b. It can be seen that the MMLCD estimates do not suffer from coalescence, and the quality of the track output is significantly improved.

*Scenario 2: Two Targets:* In this scenario, we compare the MMSE estimate with the MMOSPA, MMKD, and MMLCD estimate extracted from the same Gaussian mixture for (1). Fig. 5a depicts the estimation results of a MHT tracker for two targets. The plotted point estimates for the target positions are the MMSE estimates. In this scenario, the targets are closely-spaced such that the track estimates collapse into one position. In Fig. 5d, Fig. 5c, and Fig. 5c the same run is plotted, however, now the MMOSPA, MMKD and MMLCD estimates are used. We can see that all three prevent

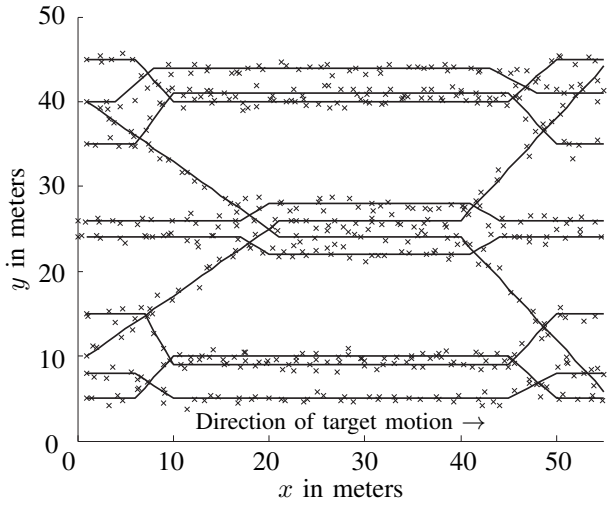



Fig. 4: Example tracks and measurements (crosses) for 10 targets. The targets move from left to right.

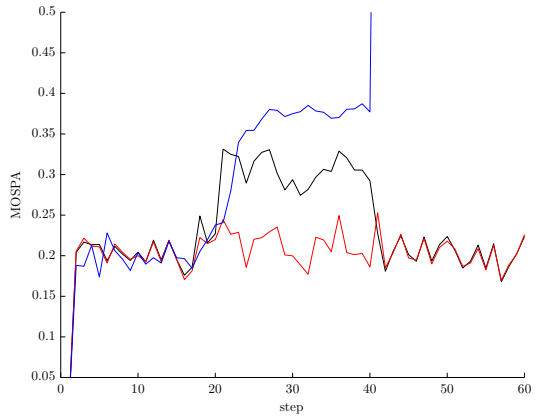



Fig. 6: Mean OSPA distance from ground truth to MMOSPA (red), MMLCD (black), and MMSE (blue) estimate. Due to track coalescence, the MOSPA error of the MMSE estimate significantly inreases after time step 40.

the tracks from coalescing and they yield comparable position estimates. However, small differences can be seen when the targets are close to each other. This impression is emphasized the Fig. 6 that shows the Mean OSPA distance of the true target position to the point estimates (MMOSPA, MMLCD and MMSE). Of course, the MMOSPA estimate gives the lowest Mean OSPA distance. However, the MMLCD estimate is close to the MMOSPA estimate. The Mean OSPA errror of the MMSE estimate increases due to track coalescence. Note that the MMLCD estimate would be the best when evaluated with the Mean LCD distance, and the MMSE estimate would be the best when evaluated with Mean Squared Error (MSE).

## X. Conclusions

As MMSE estimation implicitly considers target labels, track coalescence may occur when tracking multiple closely-spaced targets. In order overcome this issue, a distance measure that does not incorporate the target labels can be used. In this work, we have shown that the kernel distance is an intuitive and computationally attractive distance measure for extracting unlabeled point estimates. Furthermore, it also provides a systematic, theoretically well-grounded approach for extracting point estimates from PHDs. Future work consists of calculating the uncertainty for the point estimate and the incorporation of labels.

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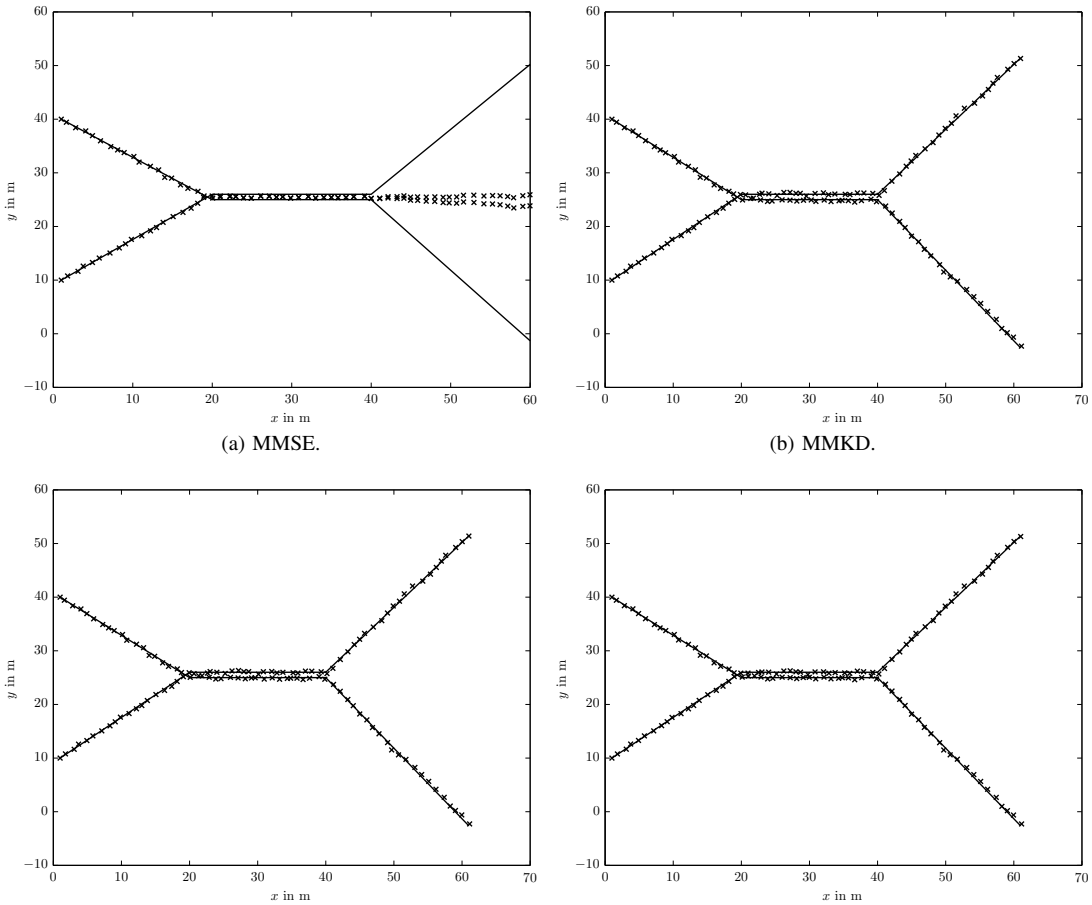
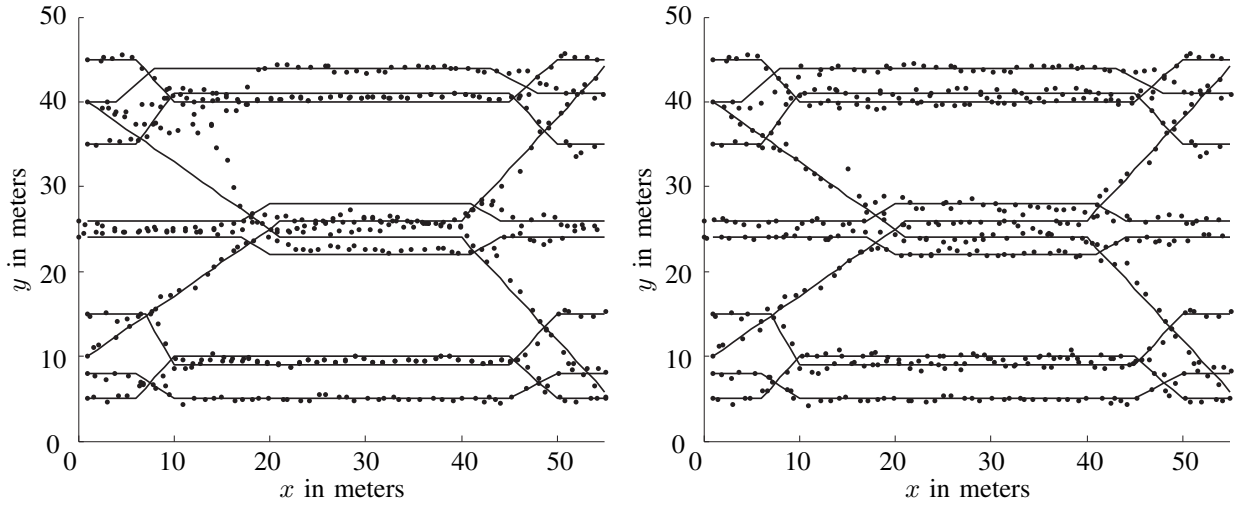



(a) MMSE. (b) MMKD.

(c) MMLCD. (d) MMOSPA.

Fig. 5: Two targets: Comparison of MMSE, MMKD, MMLCD, and MMOSPA estimates (point estimates are plottet as crosses).




(a) MMSE. (b) MMLCD.

Fig. 7: Comparison of MMSE estimate and MMLCD estimate for ten targets (point estimates are plottet as dots).