# MMOSPA-based Direction-of-Arrival Estimation for Planar Antenna Arrays

Marcus Baum, Peter Willett
Department of Electrical and Computer Engineering,
University of Connecticut, USA.
Email: baum@engineer.uconn.edu, willett@engr.uconn.edu

Uwe D. Hanebeck
Intelligent Sensor-Actuator-Systems Laboratory (ISAS),
Karlsruhe Institute of Technology (KIT), Germany.
Email: uwe.hanebeck@ieee.org

***Abstract*—This work is concerned with direction-of-arrival (DOA) estimation of narrowband signals from multiple targets using a planar antenna array. We illustrate the shortcomings of Maximum Likelihood (ML), Maximum a Posteriori (MAP), and Minimum Mean Squared Error (MMSE) estimation, issues that can be attributed to the symmetry in the likelihood function that must exist when there is no information about labeling of targets. We proffer the recently introduced concept of Minimum Mean OSPA (MMOSPA) estimation that is based on the optimal sub-pattern assignment (OSPA) metric for sets and hence inherently incorporates symmetric likelihood functions.**

## I. Introduction

The main benefits of antenna arrays [2], [8], [12]–[14], [16], [18] are an increased directional antenna gain and a flexible adjustment of the scan direction. A fundamental signal processing task for antenna arrays is direction-of-arrival (DOA) estimation of signals received from multiple targets. To this end there are many different methods. For example, spectral methods such as MUSIC [18] as well as Maximum Likelihood (ML) and Maximum a Posteriori (MAP) estimators [16] are widely used. In general ML/MAP estimators are said to provide a better statistical performance than spectral methods [8], [13], [14], [16], [18]. However, ML/MAP estimators typically come at the cost of a higher computational demand than spectral methods.

In this work, we reveal undesired coalescence effects of ML/MAP and Minimum Mean Squared Error (MMSE) estimates in case of closely-spaced targets. We show that the root of this misbehavior lies in the symmetry of the likelihood – meaning it is not possible to decide which target is which. To tackle this issue we suggest to ignore the target labels inherent to the estimation procedure. We employ the recently evolved concept of Minimum Mean OSPA (MMOSPA) estimation [3], [9], [17]. As the MMOSPA estimator is based on a metric for sets, the Optimal Sub-Pattern Assignment (OSPA) metric [15], it allows for a systematic and mathematically sound treatment of symmetric likelihood functions without coalescence effects.

MMOSPA-based DOA estimation was first proposed in [4] for two-dimensional targets. The work [5] focuses on a comparison with compressive sensing [11] in case of three-dimensional targets and a linear array. Here, we consider planar arrays and employ a recently developed exact algorithm for calculating MMOSPA estimates for particles [1]. Further, we concentrate on a detailed and intuitive discussion of the benefits of MMOSPA compared to ML and MMSE.

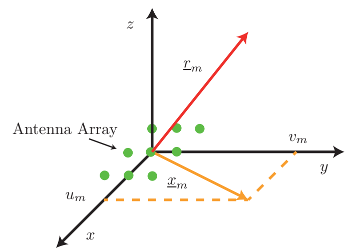



Fig. 1: Planar antenna array arranged in a regular grid. The direction vector for target $m$ is given by $\underline{r}_m$ and the corresponding vector in $uv$-space is $\underline{x}_m = \left[u_m, v_m\right]^T$.

## II. Problem Formulation

We consider an antenna array consisting of $N_p$ antennas located at $\underline{c}_1, \ldots, \underline{c}_{N_p} \in \mathbb{R}^3$ in three-dimensional space. W.l.o.g. we assume that the first antenna is located at the origin, i.e., $\underline{c}_1 = \left[0, 0, 0\right]^T$. The objective is to estimate the directions-of-arrival (DOAs) of waveforms transmitted from $N_t$ targets in three-dimensional space. We assume that the number of targets $N_t$ is given. In general, the DOAs can be specified by direction vectors $\underline{r}_1, \ldots, \underline{r}_{N_t} \in \mathbb{R}^3$ (note that a direction vector has magnitude one). As we make the common assumption that the targets are located in the upper half-space, the DOAs of targets can be represented more compactly in the two-dimensional $uv$-space, which is constructed by projecting direction vectors on the unit disc in the $xy$-plane. In this vein, the DOA of target $m$ in $uv$-space is $\underline{x}_m := \left[u_m, v_m\right]^T$, where the corresponding direction vector is $\underline{r}_m = \left[u_m, v_m, \sqrt{1 - u_m^2 - v_m^2}\right]^T$.

Each target transmits a planar wavefront impinging on the antennas. The time difference of the propagation path with respect to the origin for target $l$ is given by

$$d_{l,m}(\underline{x}_m) = \underline{c}_l^T \underline{r}_m \quad .$$

We consider narrowband signals, i.e., time-delays are phase shifts, and assume the signals to be unknown. Hence, the complex envelope of the received signal at the $i$-th antenna is modeled as (see for example [2], [8], [13], [14], [16], [18])

$$z_l = \sum_m a_l(\underline{x}_m) s_m + w_l \quad , \tag{1}$$

where

- $s_m$ denotes the unknown complex signal of target $m$ modeled as zero-mean complex Gaussian random variable,
- $a_l(\underline{x}_m) = e^{j\frac{2\pi}{\lambda}d_{l,m}(\underline{x}_m)}$ is the response of antenna $l$ to a signal with wavelength $\lambda$, and
- $w_l$ is complex zero-mean Gaussian noise.

If the received signals for all antennas are stacked to form a single vector $\underline{z} = \left[z_1, \ldots, z_{N_a}\right]^T \in \mathbb{R}^{N_a}$, we obtain the overall measurement equation

$$\underline{z} = \mathbf{A}(\underline{x})\underline{s} + \underline{w} \tag{2}$$

with stacked DOAs $\underline{x} = \left[\underline{x}_1^T, \ldots, \underline{x}_{N_t}^T\right]^T \in \mathbb{R}^{N_t}$, array response matrix $\mathbf{A}(\underline{x}) = (a_l(\underline{x}_m))_{l,m} \in \mathbb{R}^{N_a \cdot N_t}$, and stacked noise vector $\underline{w} = \left[w_1, \ldots, w_{N_a}\right]^T \in \mathbb{R}^{N_a}$. Based on (2), we can derive the likelihood function

$$p(\underline{z}\,|\,\underline{x}_k) = \mathcal{CN}(\underline{z} - \underline{0}, \mathbf{C}_w + \mathbf{A}(\underline{x})\mathbf{C}_s(\mathbf{A}(\underline{x}))^H) \ , \tag{3}$$

which is a complex Gaussian distribution with zero mean and a covariance matrix that depends on the covariance matrix $\mathbf{C}_s$ of the signals $\underline{s}$ and the joint covariance matrix $\mathbf{C}_w$ of the noise $\underline{w}$.

In this work, we assume that the prior for the target DOAs is of the form

$$p(\underline{x}) = \Pi_{m=1}^{N_t} p(\underline{x}_m) \ , \tag{4}$$

i.e., the DOAs are independent. Further, the individual priors $p(\underline{x}_m)$ are uniformly distributed on the unit disc, which essentially expresses a total lack of prior knowledge about the DOAs. Based on Bayes' rule, the posterior density for the DOAs becomes

$$p(\underline{x}\,|\,\underline{z}) = \alpha \cdot p(\underline{z}\,|\,\underline{x}_k) \cdot p(\underline{x}) \ , \tag{5}$$

where $\alpha$ is a normalization constant.

Having determined the posterior density (5) for a measurement $\underline{z}$, the question is how to extract a single point estimate for the DOAs in $\underline{x}$. In the following, we will discuss two standard point estimates, i.e., the MMSE and MAP estimate, and discuss their shortcomings for the DOAs estimation. Subsequently, we propose the MMOSPA estimate and elaborate why it is superior to the MMSE and MAP estimate in the DOAs estimation application.

## III. MMSE Estimation of DOAs

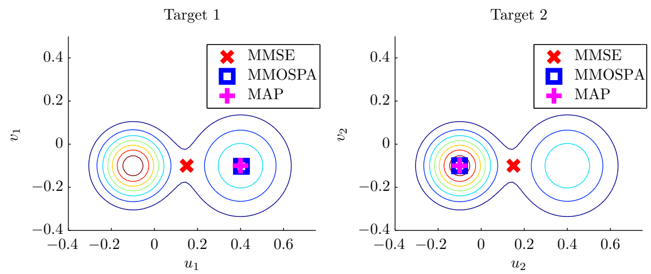
The MMSE estimate is defined as

$$\hat{\underline{x}}^{\mathrm{MMSE}} := \arg\min_{\hat{\underline{x}}} \int ||\hat{\underline{x}} - \underline{x}||^2 \, p(\underline{x}\,|\,\underline{z}) \, d\underline{x} \ . \tag{6}$$

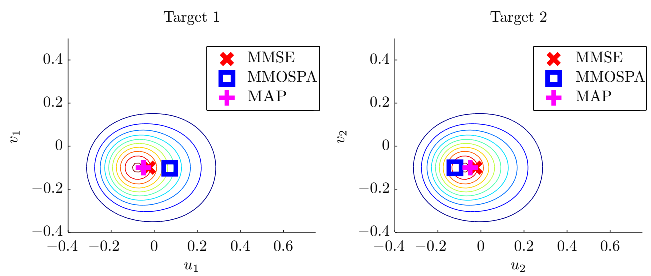
It is well known that an explicit expression for the MMSE estimate is given by the mean of $p(\underline{x}\,|\,\underline{z})$

$$\hat{\underline{x}}^{\mathrm{MMSE}} = \int \underline{x} \, p(\underline{x}\,|\,\underline{z}) \, d\underline{x} \ . \tag{7}$$

In order to illustrate the problems of the MMSE estimate for DOA estimation, it is essential to note that the received signal $\underline{z}_l$ in (1) is composed of sums representing the individual target signals, i.e., $\underline{x}_m$ and $\underline{s}_m$. Hence, we can reorder the target states without changing $\underline{z}_l$. For example, say there are two targets with




(a) Well separated targets: Marginal densities $p(\underline{x}_1)$ and $p(\underline{x}_2)$ of $p(\underline{x}) = \mathcal{N}(\underline{x} - \left[-0.1, -0.1, 0.4, -0.1\right]^T, \mathrm{diag}(\Sigma_1, \Sigma_2)) + \mathcal{N}(\underline{x} - \left[0.4, -0.1, -0.1, -0.1\right]^T, \mathrm{diag}(\Sigma_2, \Sigma_1))$.




(b) Closely-spaced targets: Marginal densities $p(\underline{x}_1)$ and $p(\underline{x}_2)$ of $p(\underline{x}) = \mathcal{N}(\underline{x} - \left[-0.1, -0.1, 0.05, -0.1\right]^T, \mathrm{diag}(\Sigma_1, \Sigma_2)) + \mathcal{N}(\underline{x} - \left[0.05, -0.1, -0.1, -0.1\right]^T, \mathrm{diag}(\Sigma_2, \Sigma_1))$.

Fig. 2: Illustration of MMSE, ML/MAP, and MMOSPA estimates for a symmetric Gaussian mixture for two targets. In the definition of the Gaussian mixture, we use the individual covariance matrices $\Sigma_1 = \mathrm{diag}(0.0204, 0.0204)$ and $\Sigma_2 = \mathrm{diag}(0.0102, 0.0102)$. Note that the two Gaussian mixtures in a) and b) only differ in their $u$-coordinates. The marginal densities for both targets coincide due to the symmetric joint density. In a) the Gaussian components (i.e., the targets) are well separated and hence, MAP and MMOSPA estimate coincide. In b) the targets are closely-spaced, so that the MAP estimate collapses as there is only one mode. The MMOSPA estimate does not coalesce. The MMSE estimate coalesces in both cases due to the symmetric joint density.

- $\underline{x}_1 = \left[0.1, 0.2\right]^T$ and $\underline{s}_1 = 2$ for target 1, and
- $\underline{x}_2 = \left[0.5, 0.3\right]^T$ and $\underline{s}_2 = 1$ for target 2.

Then the received signal $\underline{z}_l$ does not change if we switch target 1 and target 2, i.e.,

- $\underline{x}_2 = \left[0.5, 0.3\right]^T$ and $\underline{s}_2 = 1$ for target 1, and
- $\underline{x}_1 = \left[0.1, 0.2\right]^T$ and $\underline{s}_1 = 2$ for target 2.

This means that the received signal does not contain any information about the target labels; it is not possible to identify which signal comes from which target. As a consequence, the likelihood function (3) for DOA estimation is symmetric in the target states, i.e.,

$$p(\underline{z}\,|\,\underline{x}) = p(\underline{z}\,|\,P_\pi(\underline{x})) \text{ for all } \pi \in \Pi_{N_t} \ , \tag{8}$$

where $\Pi_{N_t}$ denotes all permutations of the set $\{1, \ldots, N_t\}$ and $P_\pi(\underline{y}) := [\underline{x}_{\pi(1)}^T, \ldots, \underline{x}_{\pi(N_t)}^T]^T$ permutes the single target states in $\underline{x}$ according to $\pi$.

As the prior (4) is also symmetric, the posterior density (5) is symmetric, i.e.,

$$p(\underline{x} \,|\, \underline{z}) = p(P_\pi(\underline{x}) \,|\, \underline{z}) \text{ for all } \pi \in \Pi_n \ . \tag{9}$$

The MMSE estimate from a density that is symmetric in the target states *always* coalesces the targets, i.e.,

$$\hat{\underline{x}}_1^{\text{MMSE}} = \hat{\underline{x}}_2^{\text{MMSE}} = \ldots = \hat{\underline{x}}_{N_t}^{\text{MMSE}} \ .$$

The reason is that all marginal densities of a symmetric density coincide due to

$$\begin{aligned} p(\underline{x}_m) &= \int p(\underline{x}_1, \ldots, \underline{x}_m, \ldots, \underline{x}_n) \\ &\quad d\underline{x}_1, \ldots, d\underline{x}_{m-1}, d\underline{x}_{m+1}, \ldots, d\underline{x}_n \\ &= \int p(\underline{x}_m, \underline{x}_1, \ldots, \underline{x}_{m-1}, \underline{x}_{m+1}, \ldots, \underline{x}_{N_t}) \\ &\quad d\underline{x}_1, \ldots, d\underline{x}_{m-1}, d\underline{x}_{m+1}, \ldots, d\underline{x}_{N_t} \ . \end{aligned} \tag{10}$$

An example of the MMSE estimate in case of two targets is illustrated in Fig. 2. This behavior results from the fact the MMSE estimate is unbiased, which is in general a highly desirable property, but in our case – symmetry in the targets – is unsuitable.

Note that in the DOA estimation application, it is not straightforward to calculate the MMSE estimate, i.e., the mean, of (5). Here, we follow [5], where *importance sampling* is used in order get a particle representation of (5).

## IV. ML/MAP Estimation of DOAs

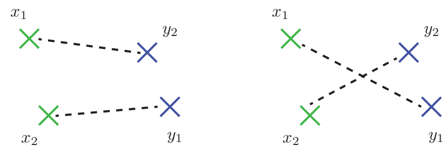
The MAP estimate

$$\hat{\underline{x}}^{\text{MAP}} := \arg\max_{\hat{\underline{x}}} p(\hat{\underline{x}} \,|\, \underline{z}) \tag{11}$$

is widely used for DOAs estimation. The MAP estimate for (5) can be obtained by, for example, the Quasi-Newton method, see [5]. Note that the MAP and ML estimate coincide for the uniform prior (4). At first blush, MAP does not suffer from coalescence issues as it chooses between the modes, see Fig. 2. However, as soon as the targets come close to together, the modes of the posterior density collapse to a single mode, and hence the MAP estimate coalesces. For example, the modes of a Gaussian mixture consisting of two one-dimensional Gaussians collapse when the absolute distance between their means is less than $2\sigma$, where $\sigma$ is the standard deviation of both Gaussians [10]. A further general disadvantage of the MAP estimate is that it tends to jitter because the MAP estimate ignores everything around the maximum of the posterior, which may change frequently (see [3] for a more detailed discussion).

## V. MMOSPA of DOAs

Recently, a novel systematic approach for estimating the states of multiple objects without considering their labels has been developed. The basic idea is to replace the squared error in the MMSE definition with a permutation invariant criterion – the *Optimal Sub-Pattern Assignment (OSPA)* [15] distance. During the last years, the OSPA distance became the standard metric for performance evaluation of multiple target trackers.




(a) Permutation $\pi_1$. (b) Permutation $\pi_2$.

Fig. 3: Illustration of the OSPA distance for two two-dimensional target state vectors $\underline{x} = [\underline{x}_1^T, \underline{x}_2^T]^T$ and $\underline{y} = [\underline{y}_1^T, \underline{y}_2^T]^T$: There are two possible permutations $\pi_1$ and $\pi_2$, i.e., $\text{OSPA}(\underline{x}, \underline{y})^2 = \frac{1}{2} \min\{||\underline{x}_1 - P_{\pi_1}(\underline{y})||^2, ||\underline{x}_2 - P_{\pi_2}(\underline{y})||^2\}$.

The OSPA distance between two vectors $\underline{x} = [\underline{x}_1^T, \ldots, \underline{x}_{N_t}^T]^T$ and $\underline{y} = [\underline{y}_1^T, \ldots, \underline{y}_{N_t}^T]^T$, which consist of $N_t$ target state vectors, is defined as

$$\text{OSPA}(\underline{x}, \underline{y})^2 := \frac{1}{N_t} \min_{\pi \in \Pi_{N_t}} ||\underline{x} - P_\pi(\underline{y})||^2 \ . \tag{12}$$

Based upon (12), the *minimum mean optimal sub-pattern assignment (MMOSPA)* can be defined [9] in analogy to the MMSE estimate, i.e.,

$$\hat{\underline{x}}^{\text{MMOSPA}} := \arg\min_{\hat{\underline{x}}} \int \text{OSPA}(\underline{x}, \hat{\underline{x}})^2 \, p(\underline{x} \,|\, \underline{z}) \, d\underline{x} \ .$$

Note that there are $N_t!$ MMOSPA estimates, as each permutation of $\hat{\underline{x}}^{\text{MMOSPA}}$ is also a MMOSPA estimate.

Due to its permutation invariance, the MMOSPA estimate does not coalesce even for closely-spaced objects, see Fig. 2, and also comes with the smoothing effects of MMSE.

Calculating MMOSPA estimates is in general difficult. However, various efficient approximations are available [6], [7], [9] and in case of a particle representation even exact efficient algorithms have recently be found [1] for relevant special cases. In order to calculate the MMOSPA estimate for (5), we first generate a particle representation of (5) using *importance sampling*. Second, the exact algorithm [1] is employed to get the MMOSPA estimate.

## VI. Evaluation

In this section, we provide a quantitative comparison of MMSE, MAP, and MMOSPA estimates from a planar antenna array consisting of 9 antennas arranged in a regular $3 \times 3$ grid with spacing $\lambda/2$, see Fig. 1 and Fig. 4. We consider two targets, where both target signals are generated according to the same complex Gaussian density. Simulations are performed for signal-to-noise ratios $10\,\text{db}$ and $13\,\text{db}$. The true DOA of target 1 is $\underline{x}_1 = \left[0.3, 0\right]^T$ in $uv$-space. The DOA of target 2 varies; it results from rotating $\underline{x}_1$ around the origin with angle $\theta$, where $\theta$ varies from 0 to $\frac{3}{4}\pi$, see Fig. 4. The MAP estimate is obtained via Quasi-Newton optimization. The MMSE and MMOSPA estimate are obtained from a particle approximation of the posterior (5), where importance sampling with 3000 samples is performed as in [4].

Fig. 5 depicts the median OSPA error for MMSE, MAP, and MMOSPA estimates for different locations of target 2. The comparison is performed with the OSPA distance as we have to compare sets. Further, we use the median error

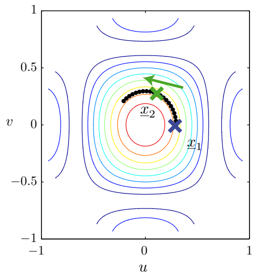
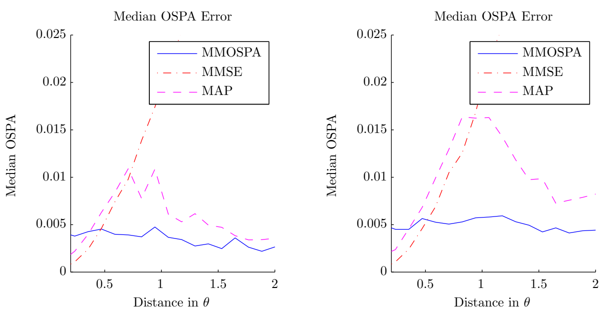



Fig. 4: Radiation pattern for the regular grid in Fig. 1 in $uv$-space and trajectory of target 2 with respect to target 1.




(a) SNR 13.

(b) SNR 10.

Fig. 5: Median OSPA error of the MMSE, MAP, and MMOSPA estimator for two targets with respect to their (angular) distance for 200 runs.

for comparison as the MAP estimate suffers from significant outliers but performs well in general. Hence, the median OSPA error draws a fairer picture of the estimation quality.

Based on Fig. 5, we can confirm the expected results from the theoretical discussions for DOAs estimation:

- The MMSE estimate is not suitable for DOAs estimation. As it always gives two DOAs at the same location (somewhere between the true DOAs), the estimation error grows with the distance of the targets.
- The MMOSPA estimate is the best and its performance is insensitive to the separation of the targets.
- For well-separated targets, the MMOSPA and MAP estimate are close together and often coincide.
- For closely-spaced targets, the MAP estimate becomes worse than the MMOSPA estimate (due to coalescence). When the targets are extremely close, a coalesced estimate can be better than the MMOSPA estimate when compared with the median OSPA metric.

## VII. Conclusion and Future Work

Based on both evaluation and intuition it appears that MMOSPA estimation is superior to ML/MAP estimation and MMSE estimation in the DOA estimation application. Admittedly MMOSPA estimation is more complex than ML/MAP estimation: while ML/MAP optimization can directly be performed based on the posterior density, it is necessary to generate a particle representation in order to extract the MMOSPA estimate. However, we have employed some recent developments to bring the computational load to a reasonable level and we hope that the reader will appreciate that MMOSPA is indeed real-time ready. Future work will focus on the development of MMOSPA estimation techniques that directly work on the posterior density, and on the effect on the estimator of the chosen state space for DOAs, i.e., $uv$-space or angular space.

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