Optimal Sequence-Based LQG Control over TCP-like Networks Subject to Random Transmission Delays and Packet Losses

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Abstract— This paper addresses the problem of sequence-based controller design for Networked Control Systems (NCS), where control inputs and measurements are transmitted over TCP-like network connections that are subject to random transmission delays and packet losses. To cope with the network effects, the controller not only sends the current control input to the actuator, but also a sequence of predicted control inputs at every time step. In this setup, we derive an optimal solution to the Linear Quadratic Gaussian (LQG) control problem and prove that the separation principle holds. Simulations demonstrate the improved performance of this optimal controller compared to other sequence-based approaches.

I. INTRODUCTION

In Networked Control Systems (NCS), components of a control loop are connected by one or more digital data networks. The applied data networks can be distinguished into real-time capable field buses, such as Interbus or Ether-CAT, which guarantee reliable data transmissions with deterministic latency, on the one hand and into general-purpose networks such as Ethernet-(TCP/UDP)/IP, or WLAN (IEEE 802.11), that have stochastic transmission characteristics on the other hand. While field buses are used in industrial control systems for decades, there is a trend towards applying general-purpose networks for several reasons. Networks like Ethernet-TCP/IP are not only cheaper than field buses, but are also based on a wide-spread, standardized, and nonproprietary communication technology [1]. Furthermore, general-purpose wireless networks allow for more flexible applications such as platooning vehicles [2].

However, general-purpose networks can be subject to time-varying transmission delays and data losses. Since these effects can strongly degrade the performance of a system ([3], [4], [5]), new control methods have been developed that take the stochastic network effects explicitly into account [6].

The approach presented in this paper belongs to a class of predictive control methods called sequence-based control, which is also referred to as networked predictive control or packet-based control [7], [8], [9], [10], [11], [12]. The main idea of the sequence-based approach is that the controller sends data packets over the network to the actuator that not contain only the current control input, but also predicted control inputs for future time instants. The predicted future
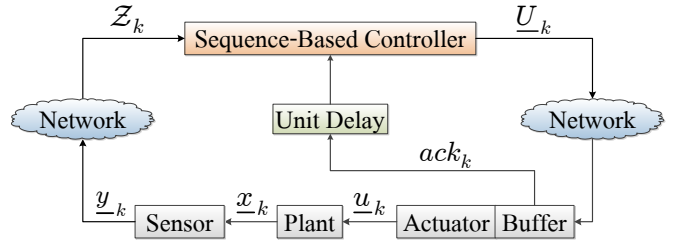
Fig. 1. Considered setup: The sequence-based controller generates control sequences $\underline{U}_k$ that are transmitted to the actuator via a TCP-like network, which provides acknowledgments (ack) for successfully transmitted data packets. The acknowledgments are available to the controller at the beginning of the next time step. At the actuator side, the control sequence containing the most recent information is hold in a buffer and the time-corresponding entry $\underline{u}_k$ of this sequence is applied to the plant. The state $\underline{x}_k$ of the plant is measured ($\underline{y}_k$) and sent via a network to the controller, which receives no, one, or even more than one measurement (denoted by the set $\mathcal{Z}_k$).

control inputs are stored in a buffer attached to the actuator so they can be applied in cases future data packets are delayed or lost. An assumption made by these methods is that the additional data sent over the network does not degrade the quality of the connection. For packet-based networks such as Ethernet-TCP/IP, this assumption usually holds as the process data needed for control applications is normally less than the size of a data packet.

In the literature, different approaches for the design of sequence-based controllers have been proposed. One class of approaches utilizes a nominal feedback-controller designed for the nominal system with the networks replaced by transparent connections [11], [13], [14], [15]. Using the nominal controller, future control inputs are predicted and control sequences generated. Unfortunately, even if the nominal controller is derived by an optimization method such as LQG, the synthesized sequence-based controller only yields suboptimal results.

Another line of sequence-based approaches evolves from Model Predictive Control (MPC) theory [8], [9], [16]. This is an intuitive connection, as MPC-controllers already obtain an optimized sequence of predicted control inputs over a finite horizon. However, like standard MPC, this approach approximates the stochastic closed-loop optimization problem by a much easier deterministic open-loop optimization problem. Therefore, the resulting controller is not optimal.

Based on [17], it has been shown in [12] that optimal sequence-based LQG controllers can be derived for NCS if

the data networks employ a so called TCP-like protocol. The term TCP-like[1] characterizes a data network that provides instantaneous acknowledgments for successfully transmitted data packets [18]. Yet, the approach in [12] neglects time delays by assuming that a data packet is either dropped by the network or transmitted immediately. In [19], the approach was extended so that time delays could be incorporated into the sequence-based controller design. However, the derived controller is not optimal as it approximates the actual time delays by its steady state distribution.

In this paper, we present the optimal solution to the problem addressed in [12] and [19] also in the presence of time-varying transmission delays. In detail, we derive an optimal solution for the sequence-based LQG control problem for NCS with TCP-like network connections subject to stochastic packet losses and time-varying packet delays. The optimal control law is derived by first using state augmentation to formulate the original networked system as a nonnetworked Markovian Jump Linear System (MJLS). Then, stochastic dynamic programming is applied on the MJLS. In the derivation, we prove that the separation principle also holds in the sequence-based setup, which was assumed in former work but not yet formally proven.

A. Notation

Throughout the paper, random variables are written in bold face letters ($\boldsymbol{a}$), deterministic quantities are in normal lettering (a), vector-valued quantities are underlined ($\underline{a}$), and matrices are bold face capital letters ($\mathbf{A}$). The terms $\mathbf{0}$ and $\mathbf{I}$ denote a matrix with all elements equal to zero and the identity matrix, respectively. Furthermore, the notation $\boldsymbol{a} \sim f(a)$ refers to a random variable $\boldsymbol{a}$ with probability density function $f(a)$. The notation a_k refers to the quantity a at time step k. For the set $\left\{\underline{x}_a, \underline{x}_{a+1}, \ldots, \underline{x}_b\right\}$, we use the abbreviated notation $\underline{x}_{a:b}$. The expectation operator is denoted by $\mathrm{E}\{\cdot\}$ and the Moore-Penrose pseudoinverse of a matrix $\mathbf{A}$ by $\mathbf{A}^\dagger$. The set of all natural numbers including zero is indicated with $\mathbb{N}_0$ and $\mathbb{N}_{>0}$ means $\mathbb{N}_0 \backslash \{0\}$.

B. Outline

The remainder of the paper is organized as follows. In the next section, the considered setup is introduced and the optimal control problem is formulated. The optimal control law is derived in Sec. III and compared with the approaches from [12] and [19] in a simulation of a double integrator system in Sec. IV. A summary and an outlook on future work concludes the paper.

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[1]A TCP-like network connection is only an approximation of a realistic Ethernet-TCP/IP network since it is assumed that the acknowledgments are not subject to time delays. However, the analysis of NCS with TCP-like connections gives insights into the more complex problem of controlling systems over real TCP connections or even over networks that do not provide any acknowledgments. In particular, the TCP-like case constitutes an upper performance bound for these cases.

II. SYSTEM SETUP & PROBLEM FORMULATION

The considered system setup is depicted in Fig. 1. Plant and sensor evolve according to

$$\begin{aligned} \underline{\boldsymbol{x}}_{k+1} &= \mathbf{A}\underline{\boldsymbol{x}}_k + \mathbf{B}\underline{\boldsymbol{u}}_k + \underline{\boldsymbol{w}}_k \ , \\ \underline{\boldsymbol{y}}_k &= \mathbf{C}\underline{\boldsymbol{x}}_k + \underline{\boldsymbol{v}}_k \ , \end{aligned} \tag{1}$$

where $\underline{\boldsymbol{x}}_k \in \mathbb{R}^n$ denotes the plant state, the vector $\underline{\boldsymbol{u}}_k \in \mathbb{R}^m$ the control input applied by the actuator, and $\underline{\boldsymbol{y}}_k \in \mathbb{R}^q$ the measured output. The matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, and $\mathbf{C} \in \mathbb{R}^{q \times n}$ are known. The terms $\underline{\boldsymbol{w}}_k \in \mathbb{R}^n$ and $\underline{\boldsymbol{v}}_k \in \mathbb{R}^q$ represent mutually independent, zero-mean, white noise processes with Gaussian probability distributions that are independent of network-induced effects. The initial state $\underline{\boldsymbol{x}}_0$ is random with Gaussian distribution and given by

$$\bar{x}_0 = E\left\{\underline{\boldsymbol{x}}_0\right\} \quad \text{and} \quad \overline{\mathbf{P}}_0 = \mathrm{E}\left\{\left(\underline{\boldsymbol{x}}_0 - \bar{x}_0\right)\left(\underline{\boldsymbol{x}}_0 - \bar{x}_0\right)^{\mathrm{T}}\right\} .$$

The network connections between controller and actuator (CA-link) and between sensor and controller (SC-link) are subject to time-varying delays and stochastic packet losses. By interpreting lost transmissions as transmissions with infinite time delay, we unify the description of both effects by only considering time-varying but possibly unbounded time delays. The time delays are described by random processes $\boldsymbol{\tau}_k^{CA}, \boldsymbol{\tau}_k^{SC} \in \mathbb{N}_0$ that specify how many time steps a data packet will be delayed if sent at time step k. Throughout the paper, we assume that $\boldsymbol{\tau}_k^{CA}$ and $\boldsymbol{\tau}_k^{SC}$ are white stationary processes and that their discrete probability density functions $f^{SC}(\tau_k^{SC})$ and $f^{CA}(\tau_k^{CA})$ are known. In addition, it is assumed that the components of the control loop are time-triggered, time-synchronized, and have identical cycle times. Furthermore, the employed network is capable of transmitting large time stamped data packets and uses a TCP-like protocol, i.e., acknowledgments are provided within the same time step as a sent packet was successfully transmitted.

Due to transmission delays and packet losses, it is possible that the controller receives no, one, or even more than one measurement per time step. The set of received measurements at time step $k \in \mathbb{N}_{>0}$ is defined as the set

$$\mathcal{Z}_k = \left\{\underline{\boldsymbol{y}}_m : m \in \{0, 1, \cdots, k\}, m + \tau_m^{SC} = k\right\} . \tag{2}$$

After processing $\mathcal{Z}_k$, the sequence-based controller generates a control sequence $\underline{U}_k$ that is sent over the network to the actuator. Entries of that sequence are denoted by $\underline{\boldsymbol{u}}_{k+m|k}$ with $m \in \{0, 1, ..., N\}$ and $N \in \mathbb{N}_0$. The index specifies that the control input is intended to be applied at time step $k+m$ and was generated at time step k. This way, a sequence of length $N+1$ generated at time step k is described by

$$\underline{U}_k = \begin{bmatrix} \underline{u}_{k|k}^{\mathrm{T}} & \underline{u}_{k+1|k}^{\mathrm{T}} & \cdots & \underline{u}_{k+N|k}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} . \tag{3}$$

Attached to the actuator is a buffer, in which the actuator stores the sequence with the most recent information among all received sequences, i.e., the sequence that was generated last (according to the time stamps). If a sequence arrives out of order, it is discarded. After updating the buffer, the actuator applies the control input of the buffered sequence

that corresponds to the current time step. If the buffer runs empty, there are different possibilities for the actuator to generate a control input [20]. In this paper, we consider the case that the actuator applies a time-invariant default control input $\underline{u}^d$. The default control input is also used to initialize the buffer. The described actuator procedure can formally be summarized by

$$\underline{u}_k = \underline{u}_{k|k-\boldsymbol{\theta}_k} \ , \tag{4}$$
$$\boldsymbol{\theta}_k = \min \left(\{ n \in \mathbb{N}_0 : m + \boldsymbol{\tau}_m^{CA} = k - n, m \in \mathbb{N}_0 \} \right.$$
$$\left. \cup \{ N+1 \} \right) \ , \tag{5}$$
$$\underline{u}_{k|k-N-1} = \underline{u}^d \ . \tag{6}$$

Remark 1 *The random variable $\boldsymbol{\theta}_k$ can be interpreted as the age of the sequence buffered in the actuator, i.e., the difference between time step of generation and actual time step. If no appropriate control input is buffered in the actuator, $\boldsymbol{\theta}_k$ is set to $N+1$ and the default control input $\underline{u}^d$ is applied according to (6).*

As we consider a TCP-like protocol, the controller can always infer which control input has been applied to the plant. This means that at time step k the controller has access to the realizations of $\boldsymbol{\theta}_{0:k-1}$ what turns out to be crucial for the separation principle to hold.

The information available to the controller at time step k is summarized by the information set $\mathcal{I}_k$ with

$$\mathcal{I}_k = \{ \bar{\underline{x}}_0, \overline{\mathbf{P}}_0, \mathcal{Z}_{1:k}, \underline{U}_{0:k-1}, \theta_{0:k-1} \} \ . \tag{7}$$

If the controller only uses the information contained in $\mathcal{I}_k$ to generate the control sequence $\underline{U}_k$ at every time step k, then the underlying control law is called admissible. In this paper, we are interested in finding an admissible control law that minimizes the cumulated linear quadratic cost function

$$C_0^K = \mathrm{E} \left\{ C_K + \sum_{k=0}^{K-1} C_k \middle| \underline{U}_{0:K-1}, \bar{\underline{x}}_0, \overline{\mathbf{P}}_0 \right\} \tag{8}$$

with stage cost

$$C_K = \underline{x}_K^{\mathrm{T}} \mathbf{Q}_K \underline{x}_K \ , \qquad C_k = \underline{x}_k^{\mathrm{T}} \mathbf{Q}_k \underline{x}_k + \underline{u}_k^{\mathrm{T}} \mathbf{R}_k \underline{u}_k \ , \tag{9}$$

where $K \in \mathbb{N}_{>0}$ is the terminal time step, $\mathbf{Q}_k$ is positive semidefinite, and $\mathbf{R}_k$ is positive definite.

Summarizing the optimal control problem, we seek to find an admissible control law that minimizes the cost (8) subject to the system equations (1), the information available (7), and the actuator logic (4) - (6).

III. DERIVATION OF THE OPTIMAL CONTROLLER

In order to derive the optimal controller, we model the system as a MJLS in Sec. III-A. Based on this model, we derive the optimal controller via stochastic dynamic programming in Sec. III-B.

A. System Modeling

As given in (4), the control input applied by the actuator at time step k depends on the random variable $\boldsymbol{\theta}_k$. In [14] and [21], it has been shown that $\boldsymbol{\theta}_k$ can be described as state of a Markov chain with transition matrix $\mathbf{T}$ according to

$$\mathbf{T} = \begin{bmatrix} p_{00} & p_{01} & 0 & 0 & \cdots & 0 \\ p_{10} & p_{11} & p_{12} & 0 & \cdots & 0 \\ p_{20} & p_{21} & p_{22} & p_{23} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & & p_{(r-1)(r)} \\ p_{r0} & p_{r1} & p_{r2} & p_{r3} & \cdots & p_{rr} \end{bmatrix} \ , \tag{10}$$

with $\quad p_{ij} = \mathrm{Prob} \left[\boldsymbol{\theta}_{k+1} = j | \boldsymbol{\theta}_k = i \right], \quad r = N + 1 \ .$

The elements of $\mathbf{T}$ in the upper right triangle are zero as $\boldsymbol{\theta}_k$ can only increase by one per time step. The remaining entries can be calculated by

$$p_{ji} = \begin{cases} 1 - \sum_{s=0}^{i} q_s & \text{for } i = j+1 \ , \\ q_i & \text{for } i \leq j < N+1 \ , \\ 1 - \sum_{s=0}^{N} q_s & \text{for } i = j = N+1 \ , \end{cases}$$

where q_s is the probability that a sequence is delayed for $s \in \mathbb{N}_0$ time steps, which can directly be derived from $f^{CA}(\tau_k^{CA})$.

To describe all relevant control inputs of former sent sequences, we introduce the vector

$$\underline{\eta}_k = \begin{pmatrix} [\underline{u}_{k|k-1}^{\mathrm{T}} & \underline{u}_{k+1|k-1}^{\mathrm{T}} & \cdots & \underline{u}_{k+N-1|k-1}^{\mathrm{T}}]^{\mathrm{T}} \\ [\underline{u}_{k|k-2}^{\mathrm{T}} & \underline{u}_{k+1|k-2}^{\mathrm{T}} & \cdots & \underline{u}_{k+N-2|k-2}^{\mathrm{T}}]^{\mathrm{T}} \\ \vdots \\ [\underline{u}_{k|k-N+1}^{\mathrm{T}} & \underline{u}_{k+1|k-N+1}^{\mathrm{T}}]^{\mathrm{T}} \\ \underline{u}_{k|k-N} \\ \underline{u}^d \end{pmatrix} \ , \tag{11}$$

with $\underline{\eta}_k \in \mathbb{R}^d$ and $d = m + m \cdot \sum_{i=1}^{N} i$. The vector $\underline{\eta}_k$ contains the default control input $\underline{u}^d$ and all control inputs of the former sent sequences $\underline{U}_{k-1}, \cdots, \underline{U}_{k-N}$ that still could be applied by the actuator either in the current time step or in the future. This is illustrated in Fig. 2, where the relevant control sequences are depicted for the case of $N = 2$.

Combining $\underline{\eta}_k$ and $\boldsymbol{\theta}_k$, the following state space model of network and actuator can be formulated

$$\underline{\eta}_{k+1} = \mathbf{F} \underline{\eta}_k + \mathbf{G} \underline{U}_k \ , \tag{12}$$
$$\underline{u}_k = \mathbf{H}_k \underline{\eta}_k + \mathbf{J}_k \underline{U}_k \ , \tag{13}$$

with

$$\mathbf{F} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \end{bmatrix} \ , \qquad \mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \ ,$$

$$\mathbf{J}_k = [\delta_{(\boldsymbol{\theta}_k, 0)} \cdot \mathbf{I} \quad \mathbf{0}] \ ,$$

$$\mathbf{H}_k = [\delta_{(\boldsymbol{\theta}_k, 1)} \cdot \mathbf{I} \quad \mathbf{0} \quad \delta_{(\boldsymbol{\theta}_k, 2)} \cdot \mathbf{I} \quad \mathbf{0} \quad \cdots \quad \delta_{(\boldsymbol{\theta}_k, N)} \cdot \mathbf{I}] \ ,$$
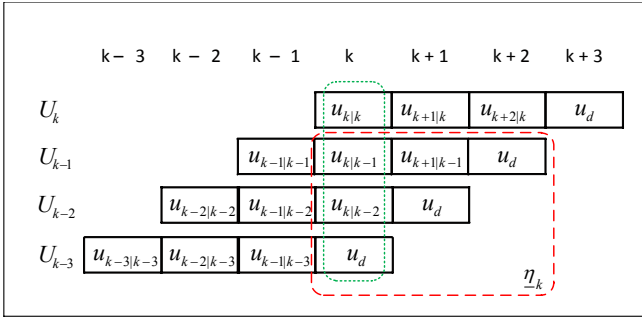
Fig. 2. Representation of control sequences $\underline{U}_{k-3}, \ldots, \underline{U}_k$, whereas control inputs corresponding to the same time step are vertically aligned. The default control $\underline{u}^d$ is added to the end of every sequence. The control inputs that could be applied by the actuator at time step k are marked by the green doted rectangle and the control inputs that are part of $\underline{\eta}_k$ are marked by the red dashed rectangle.

$$\delta_{(\boldsymbol{\theta}_k, i)} = \begin{cases} 1 & \text{if} \quad \boldsymbol{\theta}_k = i \\ 0 & \text{if} \quad \boldsymbol{\theta}_k \neq i \end{cases} .$$

Introducing the augmented state $\underline{\boldsymbol{\xi}}_k = \begin{bmatrix} \boldsymbol{x}_k^{\mathrm{T}} & \underline{\eta}_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ and combining (1), (12), and (13), it holds that

$$\underline{\boldsymbol{\xi}}_{k+1} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \cdot \mathbf{H}_k \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \underline{\boldsymbol{\xi}}_k + \begin{bmatrix} \mathbf{B} \cdot \mathbf{J}_k \\ \mathbf{G} \end{bmatrix} \underline{U}_k + \begin{pmatrix} \underline{\boldsymbol{w}}_k \\ \mathbf{0} \end{pmatrix} \quad (14)$$

$$= \widetilde{\mathbf{A}}_k \underline{\boldsymbol{\xi}}_k + \widetilde{\mathbf{B}}_k \underline{U}_k + \widetilde{\underline{\boldsymbol{w}}}_k . \quad (15)$$

Due to the stochastic parameter $\boldsymbol{\theta}_k$, the derived system model (15) is a MJLS.

B. Calculation of the Control Law

In the following, we will use the MJLS model (15) to derive the optimal control law by dynamic programming.

Remark 2 *In literature, there are several solutions to the LQG control problem of MJLS, e.g. see [22]. However, these solutions cannot be applied directly since they assume that the mode $\boldsymbol{\theta}_k$ is known without delay and that the weighting matrices $\widetilde{\mathbf{Q}}_k$ and $\widetilde{\mathbf{R}}_k$ are deterministic.*

We define the minimal expected cost-to-go J_k^* by

$$J_k^* = \min_{\underline{U}_k} \mathrm{E}\left\{ C_k + J_{k+1}^* | \mathcal{I}_k \right\} , \quad (16)$$

$$J_K^* = \mathrm{E}\left\{ C_K | \mathcal{I}_K \right\} . \quad (17)$$

According to dynamic programming theory [23], it holds that

$$J_0^* = \min_{\underline{U}_{0:K-1}} C_0^K ,$$

where C_0^K denotes the expected cumulated cost (8). To use (16) and (17), we have to express the stage cost (9) in terms of the augmented system state $\underline{\boldsymbol{\xi}}_k$. It holds, that

$$C_K = \boldsymbol{x}_K^{\mathrm{T}} \mathbf{Q}_K \boldsymbol{x}_K = \underline{\boldsymbol{\xi}}_K^{\mathrm{T}} \begin{bmatrix} \mathbf{Q}_K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \underline{\boldsymbol{\xi}}_K = \underline{\boldsymbol{\xi}}_K^{\mathrm{T}} \widetilde{\mathbf{Q}}_K \underline{\boldsymbol{\xi}}_K, \quad (18)$$

$$C_k = \boldsymbol{x}_k^{\mathrm{T}} \mathbf{Q}_k \boldsymbol{x}_k + \left(\mathbf{H}_k \underline{\eta}_k + \mathbf{J}_k \underline{U}_k \right)^{\mathrm{T}} \mathbf{R}_k \left(\mathbf{H}_k \underline{\eta}_k + \mathbf{J}_k \underline{U}_k \right)$$

$$= \begin{pmatrix} \boldsymbol{x}_k \\ \underline{\eta}_k \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{Q}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_k^{\mathrm{T}} \mathbf{R}_k \mathbf{H}_k \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_k \\ \underline{\eta}_k \end{pmatrix} + \underline{U}_k^{\mathrm{T}} \mathbf{J}_k^{\mathrm{T}} \mathbf{R}_k \mathbf{J}_k \underline{U}_k$$

$$= \underline{\boldsymbol{\xi}}_k^{\mathrm{T}} \widetilde{\mathbf{Q}}_k \underline{\boldsymbol{\xi}}_k + \underline{U}_k^{\mathrm{T}} \widetilde{\mathbf{R}}_k \underline{U}_k ,$$

where we defined

$$\widetilde{\mathbf{R}}_k = \mathbf{J}_k^{\mathrm{T}} \mathbf{R}_k \mathbf{J}_k ,$$

$$\widetilde{\mathbf{Q}}_K = \begin{bmatrix} \mathbf{Q}_K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} , \qquad \widetilde{\mathbf{Q}}_k = \begin{bmatrix} \mathbf{Q}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_k^{\mathrm{T}} \mathbf{R}_k \mathbf{H}_k \end{bmatrix}$$

and dropped the term $\mathbf{H}_k^{\mathrm{T}} \mathbf{R}_k \mathbf{J}_k$ as together with (16) it always holds that $\mathrm{E}\left\{ \mathbf{H}_k^{\mathrm{T}} \mathbf{R}_k \mathbf{J}_k \right\} = 0$. Starting at time step K, the minimal expected cost-to-go are directly given by (17) and (18). Introducing the definition $\mathbf{K}_K = \widetilde{\mathbf{Q}}_K$, it holds

$$J_K^* = \mathrm{E}\{ \underline{\boldsymbol{\xi}}_K^{\mathrm{T}} \widetilde{\mathbf{Q}}_K \underline{\boldsymbol{\xi}}_K | \mathcal{I}_K \} = \mathrm{E}\left\{ \underline{\boldsymbol{\xi}}_K^{\mathrm{T}} \mathbf{K}_K \underline{\boldsymbol{\xi}}_K | \mathcal{I}_K \right\} .$$

Using (16), the minimal expected cost-to-go at time step $K - 1$ can be computed by

$$J_{K-1}^* = \min_{\underline{U}_{K-1}} \mathrm{E}\left\{ \underline{\boldsymbol{\xi}}_{K-1}^{\mathrm{T}} \widetilde{\mathbf{Q}}_{K-1} \underline{\boldsymbol{\xi}}_{K-1} + \underline{U}_{K-1}^{\mathrm{T}} \widetilde{\mathbf{R}}_{K-1} \underline{U}_{K-1} \right.$$

$$\left. + \; J_K^* | \mathcal{I}_{K-1} \right\}$$

$$= \mathrm{E}\left\{ \underline{\boldsymbol{\xi}}_{K-1}^{\mathrm{T}} \widetilde{\mathbf{Q}}_{K-1} \underline{\boldsymbol{\xi}}_{K-1} | \mathcal{I}_{K-1} \right\}$$

$$+ \min_{\underline{U}_{K-1}} \left[\mathrm{E}\left\{ \underline{U}_{K-1}^{\mathrm{T}} \widetilde{\mathbf{R}}_{K-1} \underline{U}_{K-1} + \underline{\boldsymbol{\xi}}_K^{\mathrm{T}} \mathbf{K}_K \underline{\boldsymbol{\xi}}_K | \mathcal{I}_{K-1} \right\} \right]$$

$$\tag{19}$$

$$= \mathrm{E}\left\{ \underline{\boldsymbol{\xi}}_{K-1}^{\mathrm{T}} \left(\widetilde{\mathbf{Q}}_{K-1} + \widetilde{\mathbf{A}}_{K-1}^{\mathrm{T}} \mathbf{K}_K \widetilde{\mathbf{A}}_{K-1} \right) \underline{\boldsymbol{\xi}}_{K-1} | \mathcal{I}_{K-1} \right\}$$

$$+ \min_{\underline{U}_{K-1}} \left[\underline{U}_{K-1}^{\mathrm{T}} \mathrm{E}\left\{ \widetilde{\mathbf{R}}_{K-1} + \widetilde{\mathbf{B}}_{K-1}^{\mathrm{T}} \mathbf{K}_K \widetilde{\mathbf{B}}_{K-1} | \mathcal{I}_{K-1} \right\} \underline{U}_{K-1} \right.$$

$$+ \; 2 \cdot \mathrm{E}\left\{ \underline{\boldsymbol{\xi}}_{K-1}^{\mathrm{T}} | \mathcal{I}_{K-1} \right\} \mathrm{E}\left\{ \widetilde{\mathbf{A}}_{K-1}^{\mathrm{T}} \mathbf{K}_K \widetilde{\mathbf{B}}_{K-1} | \mathcal{I}_{K-1} \right\} \underline{U}_{K-1} \Bigg]$$

$$+ \mathrm{E}\left\{ \widetilde{\underline{\boldsymbol{w}}}_{K-1}^{\mathrm{T}} \mathbf{K}_K \widetilde{\underline{\boldsymbol{w}}}_{K-1} | \mathcal{I}_{K-1} \right\} , \tag{20}$$

where we used $\mathrm{E}\{ \mathrm{E}\{ g(\underline{\boldsymbol{\xi}}_{k+1}) | \mathcal{I}_{k+1} \} | \mathcal{I}_k \} = \mathrm{E}\{ g(\underline{\boldsymbol{\xi}}_{k+1}) | \mathcal{I}_k \}$ for any function $g(\cdot)$. Furthermore, we used the fact that if $\mathcal{I}_{K-1}$ is given, then $\underline{\boldsymbol{\xi}}_{K-1}$ is conditionally independent of $\boldsymbol{\theta}_{K-1}$ and, therefore, of $\widetilde{\mathbf{A}}_{K-1}, \widetilde{\mathbf{B}}_{K-1}$, and $\mathbf{K}_K$. Differentiation of (20) with respect to $\underline{U}_{K-1}$ and setting equal to zero yields

$$\underline{U}_{K-1} = - \left(\mathrm{E}\left\{ \widetilde{\mathbf{R}}_{K-1} + \widetilde{\mathbf{B}}_{K-1}^{\mathrm{T}} \mathbf{K}_K \widetilde{\mathbf{B}}_{K-1} | \mathcal{I}_{K-1} \right\} \right)^{\dagger}$$

$$\times \mathrm{E}\left\{ \widetilde{\mathbf{B}}_{K-1}^{\mathrm{T}} \mathbf{K}_K \widetilde{\mathbf{A}}_{K-1} | \mathcal{I}_{K-1} \right\} \mathrm{E}\left\{ \underline{\boldsymbol{\xi}}_{K-1} | \mathcal{I}_{K-1} \right\} . \quad (21)$$

Remark 3 *In (21), we have used the Moore-Penrose pseudoinverse instead of the regular inverse as the expression $\mathbf{M} = \mathrm{E}\left\{ \widetilde{\mathbf{R}}_{K-1} + \widetilde{\mathbf{B}}_{K-1}^{\mathrm{T}} \mathbf{K}_K \widetilde{\mathbf{B}}_{K-1} | \mathcal{I}_{K-1} \right\}$ is in general positive semidefinite. This results from two facts: 1) if the network has a minimum latency that is longer than a time step, then the first control inputs of each sequence will never be applied and 2) the last N control sequences $\underline{U}_{K-N:K-1}$ contain control inputs (such as $\underline{u}_{K+1|K-1}$) that are supposed to be applied after the terminal time K. Therefore, the minimization problem is not well defined. One way to cope with this problem is to exclude the corresponding control inputs from the system equations. The same result is obtained by using the Moore-Penrose pseudoinverse instead of the regular inverse, since the kernel of $\mathbf{M}^{\dagger}\mathbf{M}$ is equal to the subspace corresponding to the dimensions of the undefined entries of U_k.*

Using (20) in (21) gives

$$J_{K-1}^* = \mathrm{E}\left\{\widetilde{\underline{w}}_{K-1}^{\mathrm{T}}\mathbf{K}_K\widetilde{\underline{w}}_{K-1}|\mathcal{I}_{K-1}\right\}$$
$$+ \mathrm{E}\left\{\underline{\boldsymbol{\xi}}_{K-1}^{\mathrm{T}}\mathbf{K}_{K-1}\underline{\boldsymbol{\xi}}_{K-1}|\mathcal{I}_{K-1}\right\}$$
$$+ \mathrm{E}\left\{\underline{e}_{K-1}^{\mathrm{T}}\mathbf{P}_{K-1}\underline{e}_{K-1}|\mathcal{I}_{K-1}\right\} ,$$

with $\quad \underline{e}_k = \underline{\boldsymbol{\xi}}_k - \mathrm{E}\left\{\underline{\boldsymbol{\xi}}_k|\mathcal{I}_k\right\} ,$

$$\mathbf{K}_{K-1} = \mathrm{E}\left\{\widetilde{\mathbf{Q}}_{K-1} + \widetilde{\mathbf{A}}_{K-1}^{\mathrm{T}}\mathbf{K}_K\widetilde{\mathbf{A}}_{K-1}|\mathcal{I}_{K-1}\right\} - \mathbf{P}_{K-1} ,$$

$$\mathbf{P}_{K-1} = \mathrm{E}\left\{\widetilde{\mathbf{A}}_{K-1}^{\mathrm{T}}\mathbf{K}_K\widetilde{\mathbf{B}}_{K-1}|\mathcal{I}_{K-1}\right\}$$
$$\times \left(\mathrm{E}\left\{\widetilde{\mathbf{R}}_{K-1} + \widetilde{\mathbf{B}}_{K-1}^{\mathrm{T}}\mathbf{K}_K\widetilde{\mathbf{B}}_{K-1}|\mathcal{I}_{K-1}\right\}\right)^{\dagger}$$
$$\times \mathrm{E}\left\{\widetilde{\mathbf{B}}_{K-1}^{\mathrm{T}}\mathbf{K}_K\widetilde{\mathbf{A}}_{K-1}|\mathcal{I}_{K-1}\right\} \tag{22}$$

Considering one more time step, the minimal expected cost-to-go at time step $K-2$ can be calculated by

$$J_{K-2}^* = \min_{\underline{U}_{K-2}}\left[\mathrm{E}\left\{\underline{\boldsymbol{\xi}}_{K-2}^{\mathrm{T}}\widetilde{\mathbf{Q}}_{K-2}\underline{\boldsymbol{\xi}}_{K-2} + \underline{U}_{K-2}^{\mathrm{T}}\widetilde{\mathbf{R}}_{K-2}\underline{U}_{K-2}\right.\right.$$
$$\left.\left. + J_{K-1}^*|\mathcal{I}_{K-2},\underline{U}_{K-2}\right\}\right] \tag{23}$$
$$= \mathrm{E}\left\{\underline{\boldsymbol{\xi}}_{K-2}^{\mathrm{T}}\widetilde{\mathbf{Q}}_{K-2}\underline{\boldsymbol{\xi}}_{K-2}|\mathcal{I}_{K-2}\right\}$$
$$+ \min_{\underline{U}_{K-2}}\left[\underline{U}_{K-2}^{\mathrm{T}}\mathrm{E}\left\{\widetilde{\mathbf{R}}_{K-2}|\mathcal{I}_{K-2}\right\}\underline{U}_{K-2}\right.$$
$$\left. + \mathrm{E}\left\{\underline{\boldsymbol{\xi}}_{K-1}^{\mathrm{T}}\mathbf{K}_{K-1}\underline{\boldsymbol{\xi}}_{K-1}|\mathcal{I}_{K-2},\underline{U}_{K-2}\right\}\right]$$
$$+ \mathrm{E}\left\{\underline{e}_{K-1}^{\mathrm{T}}\mathbf{P}_{K-1}\underline{e}_{K-1}|\mathcal{I}_{K-2}\right\}$$
$$+ \mathrm{E}\left\{\widetilde{\underline{w}}_{K-1}^{\mathrm{T}}\mathbf{K}_K\widetilde{\underline{w}}_{K-1}|\mathcal{I}_{K-2}\right\} . \tag{24}$$

The term $\mathrm{E}\left\{\underline{e}_{K-1}^{\mathrm{T}}\mathbf{P}_{K-1}\underline{e}_{K-1}|\mathcal{I}_{K-2}\right\}$ represents a cost for the expected estimation error. In (24), it has been excluded from the minimization since it is independent of the control sequence $\underline{U}_{K-2}$. This is justified in the following Lemma.

Lemma 1 *The expected state estimation error*

$$\mathrm{E}\left\{\underline{e}_k^{\mathrm{T}}\mathbf{P}_k\underline{e}_k|\mathcal{I}_{k-1},\underline{U}_{k-1}\right\} ,$$

is independent of the control sequences $\underline{U}_{0:k-1}$.

Proof: The proof follows the arguments in Lemma 5.2.1 in [23], which cannot be applied directly due to the different system setup. Consider the system (15) and the following autonomous system defined by

$$\check{\underline{\boldsymbol{\xi}}}_k = \widetilde{\mathbf{A}}_{k-1}\check{\underline{\boldsymbol{\xi}}}_{k-1} + \widetilde{\underline{w}}_{k-1} , \tag{25}$$

that has same system matrix, initial conditions, noise realizations $\underline{w}_{0:k-1}$, $\underline{v}_{0:k-1}$, and network delay realizations $\boldsymbol{\tau}_{0:k-1}^{CA}$, $\boldsymbol{\tau}_{0:k-1}^{SE}$. Both systems evolve according to time-variant transformations, which are linear, so that it is possible to find (at every time step k) matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, and $\bar{\mathbf{C}}$ depending on the realizations of $\boldsymbol{\theta}_{0:k-1}$ with

$$\underline{\boldsymbol{\xi}}_k = \bar{\mathbf{A}}\underline{\boldsymbol{\xi}}_0 + \bar{\mathbf{B}}\left[\underline{U}_0^{\mathrm{T}},\dots,\underline{U}_{k-1}^{\mathrm{T}}\right]^{\mathrm{T}} + \bar{\mathbf{C}}\left[\underline{w}_0^{\mathrm{T}},\cdots,\underline{w}_{k-1}^{\mathrm{T}}\right]^{\mathrm{T}}$$
$$\check{\underline{\boldsymbol{\xi}}}_k = \bar{\mathbf{A}}\underline{\boldsymbol{\xi}}_0 + \bar{\mathbf{C}}\left[\underline{w}_0^{\mathrm{T}},\cdots,\underline{w}_{k-1}^{\mathrm{T}}\right]^{\mathrm{T}} ,$$

It holds for the expected values

$$\mathrm{E}\left\{\underline{\boldsymbol{\xi}}_k|\mathcal{I}_k\right\} = \bar{\mathbf{A}}\,\mathrm{E}\left\{\underline{\boldsymbol{\xi}}_0|\mathcal{I}_k\right\} + \bar{\mathbf{B}}\left[\underline{U}_0^{\mathrm{T}},\cdots,\underline{U}_{k-1}^{\mathrm{T}}\right]^{\mathrm{T}} ,$$
$$\mathrm{E}\left\{\check{\underline{\boldsymbol{\xi}}}_k|\mathcal{I}_k\right\} = \bar{\mathbf{A}}\,\mathrm{E}\left\{\underline{\boldsymbol{\xi}}_0|\mathcal{I}_k\right\} ,$$

where $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are known since the information vector $\mathcal{I}_k$ includes $\boldsymbol{\theta}_{0:k-1}$. The estimation errors $\underline{e}_k = \underline{\boldsymbol{\xi}}_k - \mathrm{E}\left\{\underline{\boldsymbol{\xi}}_k|\mathcal{I}_k\right\}$ and $\check{\underline{e}}_k = \check{\underline{\boldsymbol{\xi}}}_k - \mathrm{E}\left\{\check{\underline{\boldsymbol{\xi}}}_k|\mathcal{I}_k\right\}$ can be calculated by

$$\underline{e}_k = \bar{\mathbf{A}}\left(\underline{\boldsymbol{\xi}}_0 - \mathrm{E}\left\{\underline{\boldsymbol{\xi}}_0|\mathcal{I}_k\right\}\right) + \bar{\mathbf{C}}\left(\underline{w}_0^{\mathrm{T}} \quad \cdots \quad \underline{w}_{k-1}^{\mathrm{T}}\right)^{\mathrm{T}} ,$$
$$\check{\underline{e}}_k = \bar{\mathbf{A}}\left(\underline{\boldsymbol{\xi}}_0 - \mathrm{E}\left\{\underline{\boldsymbol{\xi}}_0|\mathcal{I}_k\right\}\right) + \bar{\mathbf{C}}\left(\underline{w}_0^{\mathrm{T}} \quad \cdots \quad \underline{w}_{k-1}^{\mathrm{T}}\right)^{\mathrm{T}} .$$

Since the errors are identical, $\underline{e}_k$ must be independent of $\underline{U}_{0:k-1}$ so that it holds for the error covariance

$$\mathrm{E}\left\{\underline{e}_k^{\mathrm{T}}\underline{e}_k|\mathcal{I}_{k-1},\underline{U}_{k-1}\right\} = \mathrm{E}\left\{\underline{e}_k^{\mathrm{T}}\underline{e}_k|\mathcal{Z}_{1:k-1},\boldsymbol{\theta}_{0:k-2}\right\} . \tag{26}$$

Additionally, since $\boldsymbol{\theta}_{k-2}$ is given, $\boldsymbol{\theta}_{k-1}$ and $\boldsymbol{\theta}_k$ are conditionally independent of $\mathcal{Z}_{1:k-1}$ and $\underline{U}_{0:k-1}$. It follows that

$$\mathrm{E}\left\{\mathbf{P}_k|\mathcal{I}_{k-1},\underline{U}_{k-1}\right\} = \mathrm{E}\left\{\mathbf{P}_k|\boldsymbol{\theta}_{k-2}\right\} . \tag{27}$$

Combining (26) and (27) concludes the proof. ∎

As the Lemma proves that the estimation error is independent of the control, it follows that separation holds in the considered sequence-based setup. This extends results obtained in [18], where separation was proved to hold when (only) single control inputs are sent. It is worth noting that the assumption of a TCP-like connection, i.e., that at time step k the mode $\boldsymbol{\theta}_{k-1}$ is available to the controller, is crucial for the Lemma and separation to hold.

It can be seen that the structure of (19) and (24) is the same (besides two additional terms that are independent of $\underline{U}_{0:k-1}$). Therefore, minimization over $\underline{U}_{K-2}$ will lead to a J_{K-2}^* of the same structure, so that it follows by an inductive argument that

$$\underline{U}_k = -\left(\mathrm{E}\left\{\widetilde{\mathbf{R}}_k + \widetilde{\mathbf{B}}_k^{\mathrm{T}}\mathbf{K}_{k+1}\widetilde{\mathbf{B}}_k|\mathcal{I}_k\right\}\right)^{\dagger}$$
$$\times \mathrm{E}\left\{\widetilde{\mathbf{B}}_k^{\mathrm{T}}\mathbf{K}_{k+1}\widetilde{\mathbf{A}}_k|\mathcal{I}_k\right\}\mathrm{E}\left\{\underline{\boldsymbol{\xi}}_k|\mathcal{I}_k\right\} \tag{28}$$

with

$$\mathbf{K}_k = \mathrm{E}\left\{\widetilde{\mathbf{Q}}_k + \widetilde{\mathbf{A}}_k^{\mathrm{T}}\mathbf{K}_{k+1}\widetilde{\mathbf{A}}_k|\mathcal{I}_k\right\} - \mathrm{E}\left\{\widetilde{\mathbf{A}}_k^{\mathrm{T}}\mathbf{K}_{k+1}\widetilde{\mathbf{B}}_k|\mathcal{I}_k\right\}$$
$$\times \left(\mathrm{E}\left\{\widetilde{\mathbf{R}}_k + \widetilde{\mathbf{B}}_k^{\mathrm{T}}\mathbf{K}_{k+1}\widetilde{\mathbf{B}}_k|\mathcal{I}_k\right\}\right)^{\dagger}\mathrm{E}\left\{\widetilde{\mathbf{B}}_k^{\mathrm{T}}\mathbf{K}_{k+1}\widetilde{\mathbf{A}}_k|\mathcal{I}_k\right\} . \tag{29}$$

The expected values in (28) and (29) of the matrices can be calculated by explicitly conditioning on $\boldsymbol{\theta}_{k-1} = j$. This is possible as $\boldsymbol{\theta}_{k-1}$ is part of the information set $\mathcal{I}_k$. Therefore, the control law can be written as

$$\underline{U}_k = \mathbf{L}_k\,\mathrm{E}\left\{\underline{\boldsymbol{\xi}}_k|\mathcal{I}_k\right\} , \tag{30}$$

with $\quad \mathbf{L}_k = -\left[\sum_{i=0}^{N+1} p_{ji}\left(\widetilde{\mathbf{R}}_{|i} + \widetilde{\mathbf{B}}_{|i}^{\mathrm{T}}\mathrm{E}\left\{\mathbf{K}_{k+1}|\boldsymbol{\theta}_k = i\right\}\widetilde{\mathbf{B}}_{|i}\right)\right]^{\dagger}$

$$\times \left[\sum_{i=0}^{N+1} p_{ji}\widetilde{\mathbf{B}}_{|i}^{\mathrm{T}}\mathrm{E}\left\{\mathbf{K}_{k+1}|\boldsymbol{\theta}_k = i\right\}\widetilde{\mathbf{A}}_{|i}\right] , \tag{31}$$

$$\mathrm{E}\left\{\mathbf{K}_k|\boldsymbol{\theta}_{k-1}=j\right\}$$

$$=\left[\sum_{i=0}^{N+1}p_{ji}\left(\widetilde{\mathbf{Q}}_{|i}+\widetilde{\mathbf{A}}_{|i}^{\mathrm{T}}\,\mathrm{E}\left\{\mathbf{K}_{k+1}|\boldsymbol{\theta}_k=i\right\}\widetilde{\mathbf{A}}_{|i}\right)\right]$$

$$-\left[\sum_{i=0}^{N+1}p_{ji}\widetilde{\mathbf{A}}_{|i}^{\mathrm{T}}\,\mathrm{E}\left\{\mathbf{K}_{k+1}|\boldsymbol{\theta}_k=i\right\}\widetilde{\mathbf{B}}_{|i}\right]$$

$$\times\left[\sum_{i=0}^{N+1}p_{ji}\left(\widetilde{\mathbf{R}}_{|i}+\widetilde{\mathbf{B}}_{|i}^{\mathrm{T}}\,\mathrm{E}\left\{\mathbf{K}_{k+1}|\boldsymbol{\theta}_k=i\right\}\widetilde{\mathbf{B}}_{|i}\right)\right]^{\dagger}$$

$$\times\left[\sum_{i=0}^{N+1}p_{ji}\widetilde{\mathbf{B}}_{|i}^{\mathrm{T}}\,\mathrm{E}\left\{\mathbf{K}_{k+1}|\boldsymbol{\theta}_k=i\right\}\widetilde{\mathbf{A}}_{|i}\right]\,,\tag{32}$$

where the notation $\mathbf{X}_{|i}$, with $i\in\mathbb{N}_0$, refers to the matrix $\mathbf{X}$ (dependent of $\boldsymbol{\theta}_k$), where $\boldsymbol{\theta}_k$ is set to i. The terms p_{ji} indicate the elements of transition matrix $\mathbf{T}$ given by (10).

The results derived above can be summarized as follows:

Theorem 1 *Consider the problem of finding an admissible control law with given sequence length N according to* (3) *that minimizes the cost* (8) *subject to the system equations* (1), *the information available to the controller* (7), *and the actuator logic* (4) - (6). *Then,*

1) *as in standard LQG control, the separation principle holds, i.e., the optimal control law at time step k can be separated into a) an estimator that calculates the conditional expectation* $\mathrm{E}\left\{\underline{\boldsymbol{\xi}}_k|\mathcal{I}_k\right\}$ *and b) into a controller that utilizes the optimal feedback matrix $\mathbf{L}_k$,*
2) *the optimal control law is linear in the conditional expectation of the augmented state, i.e., $U_k=\mathbf{L}_k\,\mathrm{E}\left\{\underline{\boldsymbol{\xi}}_k|\mathcal{I}_k\right\}$, and*
3) *the optimal feedback matrix $\mathbf{L}_k$ can be calculated by* (31), *whereas the matrix* $\mathrm{E}\left\{\mathbf{K}_{k+1}|\boldsymbol{\theta}_k=i\right\}$ *is obtained by the recursion* (32), *which is evolving backwards in time, with initial condition* $\mathrm{E}\left\{\mathbf{K}_K|\boldsymbol{\theta}_{K-1}=i\right\}=\widetilde{\mathbf{Q}}_K$.

The conditional expectation $\mathrm{E}\left\{\underline{\boldsymbol{\xi}}_k|\mathcal{I}_k\right\}=\mathrm{E}\left\{\underline{\boldsymbol{x}}_k|\mathcal{I}_k\right\}$ is equal to the minimum mean squared error (MMSE) estimate of the state $\underline{\boldsymbol{x}}_k$. In the literature, results on the MMSE estimator in presence of measurement delays and losses are available, see e.g., [24] and [25]. It was pointed out in [24] that the optimal estimator is a time-varying Kalman filter that is extended by a buffer to store old measurements. The filter is of finite dimension (and, therefore, can be implemented) when the required memory of the buffer is finite. This is the case when every measurement that is not lost, arrives within a maximum delay time. In practice, however, it is computationally inefficient to process measurements that have been delayed for a very long time. Therefore, the length of the buffer should be interpreted as a design parameter.

IV. SIMULATIONS

In this section, we compare the performance of the proposed optimal controller with the approaches presented in [12] and [19] by means of simulations with a double
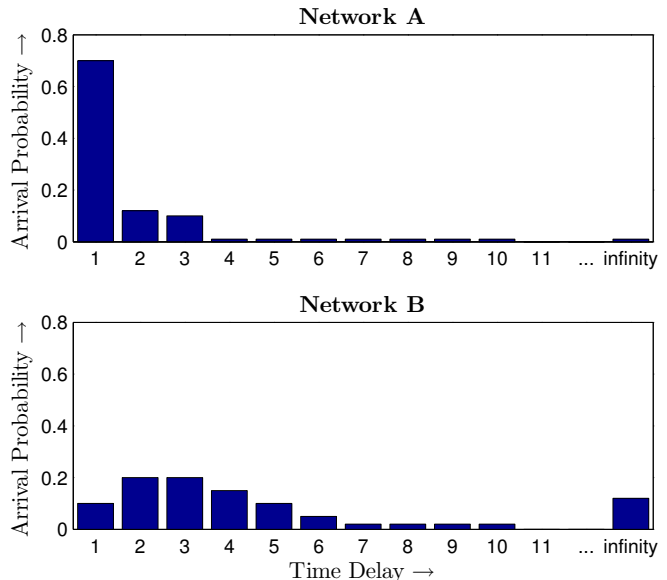
Fig. 3. Probability density functions of the arrival probability of data packets over time delays.

integrator. For simulation, the system parameters in (1) are chosen with

$$\mathbf{A}=\begin{bmatrix}1&1\\0&1\end{bmatrix},\quad\mathbf{B}=\begin{bmatrix}0\\1\end{bmatrix},\quad\mathbf{C}=\begin{bmatrix}1&0\end{bmatrix},$$

and the weighting matrices of the cost function (8), the initial condition and the noise covariances are set to

$$\mathbf{Q}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad\mathbf{R}=1\,,$$

$$\bar{x}_0=\begin{bmatrix}100\\0\end{bmatrix},\quad\bar{\mathbf{P}}_0=\begin{bmatrix}0.5^2&0\\0&0.5^2\end{bmatrix}\,,$$

$$\mathrm{E}\left\{\underline{\boldsymbol{w}}_k^{\mathrm{T}}\underline{\boldsymbol{w}}_k\right\}=\begin{bmatrix}0.1^2&0\\0&0.1^2\end{bmatrix},\quad\mathrm{E}\left\{\boldsymbol{v}_k^{\mathrm{T}}\boldsymbol{v}_k\right\}=0.2^2\,.$$

In the simulation, we use two different models of the network connections. The probability density functions of the delay distributions of both networks are depicted in Fig. 3. Network A has a better transmission quality than Network B as, first, the probability of a small time delay is significantly higher and, second, the loss probability, i.e., the probability of an infinite delay, is much smaller. We assume that the probability density function of the controller-actuator network is the same as the one of the sensor-controller network. To obtain the minimum mean squared error estimate of the state, i.e., the conditional expectation in (30), we employed the filter described in [25]. The filter is chosen so that it can process measurements with a delay of more than ten time steps and, therefore, yields the optimal state estimate. If the buffer of the actuator runs empty the actuator applies the default control input $\underline{u}^d=0$.

For each controller and network and for different length N of the control sequence, we conduct 500 Monte Carlo simulation runs over 40 time steps and calculate the average of the cumulated cost (8). The results are shown in
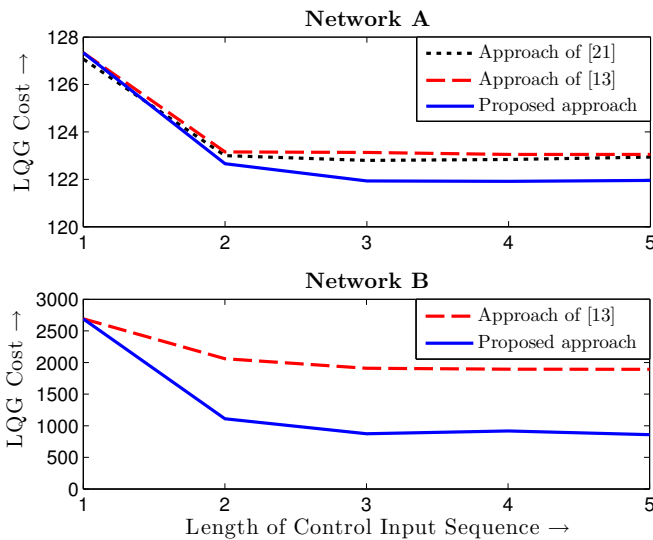
Fig. 4. Resulting averaged cumulated LQG cost for Network A and Network B for different lengths of the control sequences.

Fig. 4. The proposed controller leads to the lowest cost for both networks. For Network A, the difference between the three controllers is only small. This results from the good transmission quality of the network, in particular, from the high probability that a data packet will arrive at the actuator without any delay. This is extremely beneficial for the approach in [12], where only undelayed packets can be used by the actuator, and for the approach in [19], which is based on the state information *deemed available to the actuator* (see [19] for more details).

For Network B, the approach of [19] is not depicted as it is not able to stabilize the system, i.e., leads to cost that are several magnitudes higher than the cost of the other controllers. Compared to Network A the cumulated cost of the proposed controller and the controller from [12] are higher what results from the worse transmission quality, which unavoidably degrades the performance. However, for a sequence length of $N > 1$ the proposed approach leads to half of the cost compared to the approach of [12].

V. CONCLUSIONS

We presented an optimal solution for the sequence-based LQG control problem for NCS with TCP-like network connections. In contrast to former work, we are able to optimally consider time-varying packet delays in the sequence-based controller design.

Future work will be concerned with a derivation of stability conditions for the proposed controller. It seems reasonable to assume that the stability region is larger than the one of the approaches in [12] and [19], but this has still to be proven.

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