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Brief Paper

On combining statistical and set-theoretic estimation[1]

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Abstract

We consider state estimation based on observations which are simultaneously corrupted by a deterministic amplitude-bounded unknown bias and a possibly unbounded random process. This problem is solved by developing a combined set-theoretic and Bayesian recursive estimator. The new estimator provides a continuous transition between both concepts in that it converges to a set-theoretic estimator when the stochastic error vanishes and to a Bayesian estimator when the deterministic error vanishes. In the mixed noise case, the new estimator supplies solution *sets* defined by bounds that are uncertain in a statistical sense.

Keywords: Estimation theory; Filtering techniques; Measurement noise; Mixed noise models; Bounded noise; Stochastic noise

1. Introduction

Considering unknown bias terms in state estimation received some attention in the literature. Several authors employed augmentation of the state-space model, Anderson and Moore (1979) used a bank of Kalman filters. Hochwald and Nehorai (1995) criticized state augmentation and provided an innovative approach using minimum bias priors based on ignorance. The separate estimation of state and bias has been discussed by Friedland (1969).

We introduce a new idea for state estimation from observations of several information sources that suffer from two different uncertainties simultaneously. One type of uncertainty is a deterministic but unknown error for which hard amplitude bounds are given *a priori*. The other type of uncertainty is a stochastic process with given statistics. Prior knowledge of both forms of uncertainty allows a two-fold uncertainty reduction during the observation of sample paths of the information sources. The combined Statistical and Set-Theoretic Information (SSI) filter includes the classical estimation schemes as border cases. It converges to a set-theoretic estimator when the stochastic error approaches zero and to a Bayesian estimator when the deterministic error approaches zero. In the mixed noise case, the resulting estimate is a solution set with bounds that are uncertain in a statistical sense. For an *infinite number* of observations per source, this solution set converges to the intersection of the underlying noise-free sets. A rigorous problem formulation is given in Section 2. The special case of two information sources is discussed in Section 3. A recursive SSI filter for an arbitrary number of information sources is introduced in Section 4. Numerical solution formulae are given for arbitrary noise densities, simplified solutions arise for the case of Gaussian densities. In Section 5 numerical examples in the context of mobile robot localization are presented to clarify the conveyed concepts. While this paper is limited to the scalar case, Section 6 provides some ideas for generalization to higher dimensions.

2. Problem formulation

We consider the problem of estimating a static state x from multiple measurement streams S_i, $i = 1, \dots, N$, where measurements become available at time instants t_k. In general, measurements are simultaneously corrupted by both deterministic and stochastic errors. For example, when measuring a distance with several range sensors,

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the location of each sensor is uncertain in a deterministic sense, while each range measurement simultaneously suffers from stochastic errors.

Problems of this kind are usually approached by pure stochastic or pure deterministic estimation algorithms, i.e., Bayesian estimation or set-theoretic estimation. Application of these algorithms requires either neglecting one type of error or approximating it via the other type. Besides being theoretically incorrect, applying Bayesian estimation in the presence of deterministic uncertainties yields overoptimistic estimates. This is due to the fact that for an infinity number of measurements the estimate converges towards a biased estimate while reporting a vanishing uncertainty. On the other hand, stochastic errors with long-tailed densities pose a serious problem for set-theoretic estimators. Neglecting these errors may cause inconsistencies; approximating the errors by sets results in rather conservative estimates.

Here, we provide an estimator for the state x, which gives theoretically correct results in the presence of both error types. Both errors are assumed to be additive, because this is the most important case. The first error is of deterministic type, i.e., constant and unknown. It is bounded in amplitude by a set, which is an interval for the scalar case. The second error is represented by a discrete-time, zero-mean, possibly colored stochastic process SP_i with known statistics. The stochastic processes for different sources are mutually independent. For the scalar case, the measurement equation may be written as[2]

$$Z_i^k = x + {}^i e_\mathrm{d} + {}^i E_\mathrm{s}^k \tag{1}$$

for each measurement stream S_i, $i = 1, \ldots, N$. ${}^i e_\mathrm{d}$ denotes the deterministic error of measurement stream S_i, which is bounded by an interval of width b_i according to

$${}^i e_\mathrm{d} \in \left[-\frac{b_i}{2}, \frac{b_i}{2} \right].$$

The stochastic error at time t_k is denoted by ${}^i E_\mathrm{s}^k$, with ${}^i E_\mathrm{s}^k \sim \mathrm{SP}_i$. The estimate of state x at time t_k is based on the observations Z_i^l, $l \le k$, of the measurement streams S_i, $i = 1, \ldots, N$, which were available up to time t_k. This estimate is of course not a point estimate, but a set estimate where the set bounds are uncertain in a statistical sense.

3. Two information sources

In this section, the special case of two independent information sources S_i, $i = 1, 2$, is considered. Each source is characterized by a priori knowledge on bounds b_i of the deterministic and constant offset and noise densities f_i^k at time k. In what follows, we derive the joint density for the left and right bound of the resulting interval estimator. Analytical results are given for the marginal densities in case of Gaussian densities f_i^k.

3.1. *Arbitrary noise densities*

Let $\hat{X}_i^k$ be an estimator of $x + {}^i e_\mathrm{d}$ with density $\hat{f}_i^k$. The joint density of the independent random quantities $\hat{X}_i^k$, $i = 1, 2$, is then given by $\hat{f}_{12}^k = \hat{f}_1^k \hat{f}_2^k$. In the next step, the additional prior knowledge on the bounds b_i of the deterministic error ${}^i e_\mathrm{d}$ is used to define the new estimators $\hat{\hat{X}}_i^k$, $i = 1, 2$. This is done by eliminating those regions of the joint density $\hat{f}_{12}^k$ for which

$$|\hat{x}_2^k - \hat{x}_1^k| > \tfrac{1}{2}(b_1 + b_2)$$

holds.

The resulting normalized joint density $\hat{\hat{f}}_{12}^k$ of the new estimators $\hat{\hat{X}}_i^k$, $i = 1, 2$, is thus given by

$$\hat{\hat{f}}_{12}^k = \begin{cases} \hat{f}_{12}^k / {}^2C^k & \text{for } |\hat{\hat{x}}_2^k - \hat{\hat{x}}_1^k| \le \tfrac{1}{2}(b_1 + b_2), \\ 0 & \text{elsewhere.} \end{cases} \tag{2}$$

${}^2C^k$ is a normalizing constant, which accounts for the eliminated parts of the original density $\hat{f}_{12}^k$.

Since $\hat{\hat{X}}_i^k$, $i = 1, 2$, are estimators for the midpoint of intervals of width b_i, $i = 1, 2$, respectively (see Fig. 1) an interval estimator combining the information of both estimators $\hat{\hat{X}}_i^k$, $i = 1, 2$, is found by stochastic intersection. The left and right bounds ${}^2L^k$, ${}^2R^k$ of the interval estimator at time k are thus defined as

$${}^2L^k = \max\left(\hat{\hat{X}}_1^k - \frac{b_1}{2}, \hat{\hat{X}}_2^k - \frac{b_2}{2} \right), \tag{3}$$

$${}^2R^k = \min\left(\hat{\hat{X}}_1^k + \frac{b_1}{2}, \hat{\hat{X}}_2^k + \frac{b_2}{2} \right). \tag{4}$$

For calculating the joint distribution ${}^2F_{LR}^k(l, r)$ of ${}^2L^k$, ${}^2R^k$, Eqs. (2)–(4) provide constraints for $\hat{\hat{x}}_1^k$, $\hat{\hat{x}}_2^k$. From Eq. (2), we deduce

$$\hat{\hat{x}}_2^k - \hat{\hat{x}}_1^k \le \tfrac{1}{2}(b_1 + b_2), \qquad \hat{\hat{x}}_1^k - \hat{\hat{x}}_2^k \le \tfrac{1}{2}(b_1 + b_2). \tag{5}$$

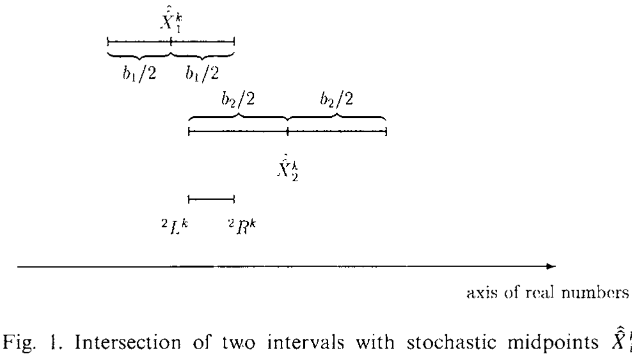

Fig. 1. Intersection of two intervals with stochastic midpoints $\hat{\hat{X}}_i^k$, $i = 1, 2$, and a priori known widths b_i, $i = 1, 2$. The resulting interval bounds ${}^2L^k$, ${}^2R^k$ are random variables.

[2] Capital letters are used for random variables or processes, small letters denote specific realizations or deterministic quantities.

Eq. (3) provides the two inequalities

$$\hat{x}_1^k \le l + \frac{b_1}{2}, \qquad \hat{x}_2^k \le l + \frac{b_2}{2}, \tag{6}$$

which must be *simultaneously* fulfilled. In contrast, Eq. (4) gives the two inequalities

$$\hat{x}_1^k \le r - \frac{b_1}{2}, \qquad \hat{x}_2^k \le r - \frac{b_2}{2}, \tag{7}$$

where *at least one* inequality must be fulfilled. Hence, these six inequalities define the shaded region depicted in Fig. 2. Integration of $\hat{f}_{12}^k$ over this region gives

$$^2f_{LR}^k(l,r) = \frac{1}{^2C^k}$$

$$\begin{cases} \Big[\hat{f}_1^k\left(r - \frac{b_1}{2}\right)\hat{f}_2^k\left(l + \frac{b_2}{2}\right) + \hat{f}_1^k\left(l + \frac{b_1}{2}\right)\hat{f}_2^k\left(r - \frac{b_2}{2}\right) \\ \quad + \delta(l + b_1 - r)\hat{f}_1^k\left(l + \frac{b_1}{2}\right)\int_{z = l + b_1 - b_2/2}^{l + b_2/2} \hat{f}_2^k(z)\,dz\Big] \\ \quad \text{for } l \le r \le l + b_1, \\ 0 \quad \text{elsewhere,} \end{cases} \tag{9}$$

where $\delta(x)$ denotes Dirac's impulse function.

$$^2F_{LR}^k(l,r) = \begin{cases} 0 & \text{for } r \le l, \\ \frac{1}{^2C^k}\Big\{\int_{\hat{x}_1^k = -\infty}^{l - b_1/2}\int_{\hat{x}_2^k = \hat{x}_1^k - (b_1 + b_2)/2}^{\hat{x}_1^k + (b_1 + b_2)/2} \hat{f}_1^k(\hat{x}_1^k)\hat{f}_2^k(\hat{x}_2^k)\,d\hat{x}_2^k\,d\hat{x}_1^k \\ \quad + \int_{\hat{x}_1^k = l - b_1/2}^{r - b_1/2}\int_{\hat{x}_2^k = \hat{x}_1^k - (b_1 + b_2)/2}^{l + b_2/2} \hat{f}_1^k(\hat{x}_1^k)\hat{f}_2^k(\hat{x}_2^k)\,d\hat{x}_2^k\,d\hat{x}_1^k \\ \quad + \int_{\hat{x}_1^k = r - b_1/2}^{l + b_1/2}\int_{\hat{x}_2^k = \hat{x}_1^k - (b_1 + b_2)/2}^{r - b_2/2} \hat{f}_1^k(\hat{x}_1^k)\hat{f}_2^k(\hat{x}_2^k)\,d\hat{x}_2^k\,d\hat{x}_1^k\Big\} & \text{for } l \le r \le l + b_1, \\ \frac{1}{^2C^k}\Big\{\int_{\hat{x}_1^k = -\infty}^{l - b_1/2}\int_{\hat{x}_2^k = \hat{x}_1^k - (b_1 + b_2)/2}^{\hat{x}_1^k + (b_1 + b_2)/2} \hat{f}_1^k(\hat{x}_1^k)\hat{f}_2^k(\hat{x}_2^k)\,d\hat{x}_2^k\,d\hat{x}_1^k \\ \quad + \int_{\hat{x}_1^k = l - b_1/2}^{l + b_1/2}\int_{\hat{x}_2^k = \hat{x}_1^k - (b_1 + b_2)/2}^{l + b_2/2} \hat{f}_1^k(\hat{x}_1^k)\hat{f}_2^k(\hat{x}_2^k)\,d\hat{x}_2^k\,d\hat{x}_1^k\Big\} & \text{for } l + b_1 \le r \le l + b_2, \\ 1 & \text{for } r \ge l + b_2, \end{cases} \tag{8}$$

where $b_1 \le b_2$ is assumed without loss of generality.

In a second step, the joint density $^2f_{LR}^k(l,r)$ is derived by applying

$$^2f_{LR}^k(l,r) = \frac{\partial^2}{\partial l \partial r}\,{}^2F_{LR}^k(l,r).$$

Calculating the derivative of $^2F_{LR}^k(l,r)$ with respect to l gives

$$\frac{\partial\{^2F_{LR}^k(l,r)\}}{\partial l} = \begin{cases} 0 & \text{for } r < l, \\ \frac{1}{^2C^k}\Big\{\int_{\hat{x}_1^k = l - b_1/2}^{r - b_1/2} \hat{f}_1^k(\hat{x}_1^k)\,d\hat{x}_1^k\ \hat{f}_2^k\left(l + \frac{b_2}{2}\right) + \hat{f}_1^k\left(l + \frac{b_1}{2}\right)\int_{\hat{x}_2^k = l - b_2/2}^{r - b_2/2} \hat{f}_2^k(\hat{x}_2^k)\,d\hat{x}_2^k\Big\} & \text{for } l < r < l + b_1, \\ \frac{1}{^2C^k}\Big\{\hat{f}_1^k\left(l + \frac{b_1}{2}\right)\int_{\hat{x}_2^k = l - b_2/2}^{l + b_2/2} \hat{f}_2^k(\hat{x}_2^k)\,d\hat{x}_2^k + \int_{\hat{x}_1^k = l - b_1/2}^{l + b_1/2} \hat{f}_1^k(\hat{x}_1^k)\,d\hat{x}_1^k\ \hat{f}_2^k\left(l + \frac{b_2}{2}\right)\Big\} & \text{for } l + b_1 < r < l + b_2, \\ 0 & \text{for } r > l + b_2. \end{cases}$$

$\partial\{^2F_{LR}^k(l,r)\}/\partial l$ is continuous at $r = l$ and $r = l + b_2$. However, there is a jump discontinuity at $r = l + b_1$. Hence, further differentiation with respect to r yields

3.2. Discussion of border cases

Eq. (9) is the general expression for combing two estimates suffering from both stochastic and deterministic uncertainties simultaneously. Of course, for the border cases of only stochastic or only deterministic uncertainties, Eq. (9) simplifies to give the well-known classical results: Bayesian estimation and set intersection.

3.2.1. Border case: stochastic uncertainties only

In this case, we consider the limit

$$b_1 \to 0, \qquad b_2 \to 0,$$

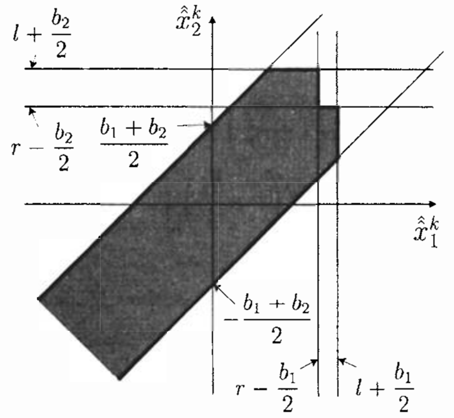

Fig. 2. Visualization aid for derivation of the joint distribution ${}^2F^k_{LR}(l, r)$.

which leads to

$$
{}^2f^k_{LR}(l,r) = \begin{cases} \dfrac{1}{{}^2C^k}\{\hat{f}^k_1(r)\hat{f}^k_2(l) + \hat{f}^k_1(l)\hat{f}^k_2(r)\} & \text{for } l = r, \\ 0 & \text{elsewhere,} \end{cases}
$$

which is equivalent to

$$
{}^2f^k_{LR}(m) = {}^2f^k_{LR}(m,m) = \frac{2}{{}^2C^k}\,\hat{f}^k_1(m)\hat{f}^k_2(m).
$$

Hence, when only stochastic uncertainties occur, the SSI-filter simplifies to the well-known Bayesian estimation result.

3.2.2. Border case: deterministic uncertainties only

In this case, we have

$$
\hat{f}_i(\hat{x}_i) = \delta(\hat{x}_i - (x + e^d_i)), \quad i = 1, 2.
$$

Hence, (9) gives

$$
{}^2f^k_{LR}(l,r) =
$$

$$
\begin{cases} \dfrac{1}{{}^2C^k}\Bigg\{\delta\left(r - \left(x + e^d_1 + \dfrac{b_1}{2}\right)\right)\delta\left(l - \left(x + e^d_2 - \dfrac{b_2}{2}\right)\right) \\ \quad + \delta\left(l - \left(x + e^d_1 - \dfrac{b_1}{2}\right)\right)\delta\left(r - \left(x + e^d_2 + \dfrac{b_2}{2}\right)\right) \\ \quad + \delta(l - (r - b_1))\,\delta\left(l - \left(x + e^d_1 - \dfrac{b_1}{2}\right)\right) \\ \quad \displaystyle\int_{z = l + b_1 - b_2/2}^{l + b_2/2} \delta(z - (x + e^d_2))\,dz\Bigg\} \quad \text{for } l \le r \le l + b_1, \\ 0 \quad \text{elsewhere.} \end{cases} \tag{10}
$$

Since $l + b_2/2 \ge x + e^d_2$, we find

$$
\int_{z = l + b_1 - b_2/2}^{l + b_2/2} \delta(z - (x + e^d_2))\,dz = \begin{cases} 0 & \text{for } b_1 = b_2, \\ 1 & \text{for } b_1 < b_2. \end{cases}
$$

Thus, Eq. (10) is equivalent to

$$
l = \max\left(x + e^d_1 - \frac{b_1}{2},\, x + e^d_2 - \frac{b_2}{2}\right),
$$

$$
r = \min\left(x + e^d_1 + \frac{b_1}{2},\, x + e^d_2 + \frac{b_2}{2}\right).
$$

Hence, when only deterministic uncertainties occur, the SSI-filter reduces to set intersection.

3.3. Gaussian noise densities

If f^k_i, $i = 1, 2$, are Gaussian, of course $\hat{f}^k_i$, $i = 1, 2$, are also Gaussian with

$$
\hat{f}^k_i(x) = \frac{1}{\sqrt{2\pi}\hat{\sigma}^k_i}\exp\left(-\frac{1}{2}\left(\frac{x - \hat{m}^k_i}{\hat{\sigma}^k_i}\right)^2\right), \tag{11}
$$

and so analytical expressions for the normalizing constant as well as the marginal densities and expected values of left and right bounds, ${}^2L^k$ and ${}^2R^k$, are obtained. The normalizing constant ${}^2C^k$ is given by

$$
\begin{aligned} {}^2C^k = {}& \operatorname{erf}\left(\frac{\hat{m}^k_1 - \hat{m}^k_2 + b_1/2 + b_2/2}{\sqrt{(\hat{\sigma}^k_1)^2 + (\hat{\sigma}^k_2)^2}}\right) \\ & - \operatorname{erf}\left(\frac{\hat{m}^k_1 - \hat{m}^k_2 - b_1/2 + b_2/2}{\sqrt{(\hat{\sigma}^k_1)^2 + (\hat{\sigma}^k_2)^2}}\right) \\ & + \operatorname{erf}\left(\frac{\hat{m}^k_2 - \hat{m}^k_1 + b_1/2 + b_2/2}{\sqrt{(\hat{\sigma}^k_1)^2 + (\hat{\sigma}^k_2)^2}}\right) \\ & - \operatorname{erf}\left(\frac{\hat{m}^k_2 - \hat{m}^k_1 + b_1/2 - b_2/2}{\sqrt{(\hat{\sigma}^k_1)^2 + (\hat{\sigma}^k_2)^2}}\right), \end{aligned} \tag{12}
$$

where $\operatorname{erf}(x)$ denotes the error function defined as

$$
\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}}\int_{y=0}^{x} \exp\left(-\frac{y^2}{2}\right) dy \tag{13}
$$

according to Papoulis (1984).

The marginal density ${}^2f^k_L(l) = \int_{r=-\infty}^{\infty} f^k_{LR}(l,r)\,dr$ of the lower bound may be expressed as

$$
\begin{aligned} {}^2f^k_L(l) = \frac{1}{{}^2C^k}\Bigg[& \hat{f}^k_1\left(l + \frac{b_1}{2}\right)\left(\operatorname{erf}\left(\frac{l + b_2/2 - \hat{m}^k_2}{\hat{\sigma}^k_2}\right) - \operatorname{erf}\left(\frac{l - b_2/2 - \hat{m}^k_2}{\hat{\sigma}^k_2}\right)\right) \\ & + \hat{f}^k_2\left(l + \frac{b_2}{2}\right)\left(\operatorname{erf}\left(\frac{l + b_1/2 - \hat{m}^k_1}{\hat{\sigma}^k_1}\right) - \operatorname{erf}\left(\frac{l - b_1/2 - \hat{m}^k_1}{\hat{\sigma}^k_1}\right)\right)\Bigg]. \end{aligned} \tag{14}
$$

The marginal density ${}^2f_R^k(r) = \int_{l=-\infty}^{\infty} f_{LR}^k(l, r)\,dl$ of the upper bound is given by

$$
\begin{aligned}
{}^2f_R^k(r) = \frac{1}{{}^2C^k}\Bigg[&\hat{f}_1^k\left(r - \frac{b_1}{2}\right)\left(\operatorname{erf}\left(\frac{r + b_2/2 - \hat{m}_2^k}{\hat{\sigma}_2^k}\right)\right.\\
&\left.- \operatorname{erf}\left(\frac{r - b_2/2 - \hat{m}_2^k}{\hat{\sigma}_2^k}\right)\right)\\
&+\hat{f}_2^k\left(r - \frac{b_2}{2}\right)\left(\operatorname{erf}\left(\frac{r + b_1/2 - \hat{m}_1^k}{\hat{\sigma}_1^k}\right)\right.\\
&\left.- \operatorname{erf}\left(\frac{r - b_1/2 - \hat{m}_1^k}{\hat{\sigma}_1^k}\right)\right)\Bigg]. \qquad (15)
\end{aligned}
$$

Expected values for lower and upper bounds are given by

$$
\begin{aligned}
E\{{}^2L^k\} = \frac{1}{{}^2C^k}\Bigg(&(\hat{\sigma}_2^k)^2\left[\mathcal{G}\left(\hat{m}_2^k - \hat{m}_1^k + \frac{b_1}{2} - \frac{b_2}{2}\right) - \mathcal{G}\left(\hat{m}_2^k - \hat{m}_1^k - \frac{b_1}{2} - \frac{b_2}{2}\right)\right]\\
&+\left(\hat{m}_2^k - \frac{b_2}{2}\right)\left[\operatorname{erf}\left(\frac{\hat{m}_2^k - \hat{m}_1^k + b_1/2 - b_2/2}{\sqrt{(\hat{\sigma}_1^k)^2 + (\hat{\sigma}_2^k)^2}}\right) - \operatorname{erf}\left(\frac{\hat{m}_2^k - \hat{m}_1^k - b_1/2 - b_2/2}{\sqrt{(\hat{\sigma}_1^k)^2 + (\hat{\sigma}_2^k)^2}}\right)\right]\\
&+(\hat{\sigma}_1^k)^2\left[\mathcal{G}\left(\hat{m}_1^k - \hat{m}_2^k - \frac{b_1}{2} + \frac{b_2}{2}\right) - \mathcal{G}\left(\hat{m}_1^k - \hat{m}_2^k - \frac{b_1}{2} - \frac{b_2}{2}\right)\right]\\
&+\left(\hat{m}_2^k - \frac{b_2}{2}\right)\left[\operatorname{erf}\left(\frac{\hat{m}_1^k - \hat{m}_2^k - b_1/2 + b_2/2}{\sqrt{(\hat{\sigma}_1^k)^2 + (\hat{\sigma}_2^k)^2}}\right) - \operatorname{erf}\left(\frac{\hat{m}_1^k - \hat{m}_2^k - b_1/2 - b_2/2}{\sqrt{(\hat{\sigma}_1^k)^2 + (\hat{\sigma}_2^k)^2}}\right)\right]\Bigg) \qquad (16)
\end{aligned}
$$

and

$$
\begin{aligned}
E\{{}^2R^k\} = \frac{1}{{}^2C^k}\Bigg(&(\hat{\sigma}_2^k)^2\left[\mathcal{G}\left(\hat{m}_2^k - \hat{m}_1^k + \frac{b_1}{2} + \frac{b_2}{2}\right) - \mathcal{G}\left(\hat{m}_2^k - \hat{m}_1^k - \frac{b_1}{2} + \frac{b_2}{2}\right)\right]\\
&+\left(\hat{m}_2^k + \frac{b_2}{2}\right)\left[\operatorname{erf}\left(\frac{\hat{m}_2^k - \hat{m}_1^k + b_1/2 + b_2/2}{\sqrt{(\hat{\sigma}_1^k)^2 + (\hat{\sigma}_2^k)^2}}\right) - \operatorname{erf}\left(\frac{\hat{m}_2^k - \hat{m}_1^k - b_1/2 + b_2/2}{\sqrt{(\hat{\sigma}_1^k)^2 + (\hat{\sigma}_2^k)^2}}\right)\right]\\
&+(\hat{\sigma}_1^k)^2\left[\mathcal{G}\left(\hat{m}_1^k - \hat{m}_2^k + \frac{b_1}{2} + \frac{b_2}{2}\right) - \mathcal{G}\left(\hat{m}_1^k - \hat{m}_2^k + \frac{b_1}{2} - \frac{b_2}{2}\right)\right]\\
&+\left(\hat{m}_2^k + \frac{b_2}{2}\right)\left[\operatorname{erf}\left(\frac{\hat{m}_1^k - \hat{m}_2^k + b_1/2 + b_2/2}{\sqrt{(\hat{\sigma}_1^k)^2 + (\hat{\sigma}_2^k)^2}}\right) - \operatorname{erf}\left(\frac{\hat{m}_1^k - \hat{m}_2^k + b_1/2 - b_2/2}{\sqrt{(\hat{\sigma}_1^k)^2 + (\hat{\sigma}_2^k)^2}}\right)\right]\Bigg), \qquad (17)
\end{aligned}
$$

with

$$\mathcal{G}(x) = \frac{\exp(-\frac{1}{2}x^2/((\hat{\sigma}_1^k)^2 + (\hat{\sigma}_2^k)^2))}{\sqrt{2\pi}\sqrt{(\hat{\sigma}_1^k)^2 + (\hat{\sigma}_2^k)^2}}. \qquad (18)$$

4. *N* information sources

The above results for two information sources are now generalized to the case of N sources. We focus attention on the derivation of the marginal densities for the upper and lower bounds, since these are of major interest for practical applications.

4.1. *Arbitrary noise densities*

We begin with the derivation of a source-recursive expression for the marginal density ${}^jf_L^k$ of the lower bound denoted by ${}^jL^k$, which contains the information from sources $1, \ldots, j$, and is defined by

$${}^jL^k = \max\left({}^{j-1}L^k,\ \hat{\hat{X}}_j^k - \frac{b_j}{2}\right). \qquad (19)$$

Hence, the joint density of ${}^{j-1}L^k$ and $\hat{\hat{X}}_j^k$ is required for calculating ${}^jL^k$. This joint density is calculated from the joint density of ${}^{j-1}L^k$, ${}^{j-1}R^k$, and $\hat{\hat{X}}_j^k$, which is given by the product

$${}^{j-1}f_{LR\hat{X}}^k(l, r, x) = \hat{\hat{f}}_j^k(x)\ {}^{j-1}f_{LR}^k(l, r),$$

because the stochastic errors of the corresponding measurement streams are assumed to be independent.

The joint density of ${}^{j-1}L^k$ and $\hat{\hat{X}}_j^k$ is then obtained by integrating over r according to

$${}^{j-1}f_{L\hat{X}}^k(l, x) = \int_{r=-\infty}^{\infty} {}^{j-1}f_{LR\hat{X}}^k(l, r, x)\,dr.$$

With Eq. (19), the marginal distribution ${}^jF_L^k$ of the lower bound is given by

$${}^jF_L^k({}^jl_k) = \int_{l=-\infty}^{{}^jl_k}\int_{x=-\infty}^{{}^jl_k + b_j/2} {}^{j-1}f_{L\hat{X}}^k(l, x)\,dx\,dl.$$

The prior knowledge on the deterministic uncertainty bound b_j may be used to formulate the following inequalities for specific realizations $\hat{\hat{x}}_j^k$, ${}^{j-1}l^k$, ${}^{j-1}r^k$:

$$\hat{\hat{x}}_j^k - \frac{b_j}{2} \le {}^{j-1}r^k, \qquad \hat{\hat{x}}_j^k + \frac{b_j}{2} \ge {}^{j-1}l^k. \qquad (20)$$

Making use of these inequalities for expressing $\hat{\hat{f}}_j^k$ in terms of $\hat{f}_j^k$ together with

$${}^j f_L^k({}^j l_k) = \frac{\partial}{\partial\, {}^j l_k}\, {}^j F_L^k({}^j l_k)$$

yields

$${}^j f_L^k({}^j l_k) = \frac{1}{{}^j C_L^k}\frac{\partial}{\partial\, {}^j l_k}\int_{l=-\infty}^{{}^j l_k}\int_{x=l-b_j/2}^{{}^j l_k+b_j/2}\int_{r=x-b_j/2}^{\infty} \times \hat{f}_j^k(x)\, {}^{j-1}f_{LR}^k(l,r)\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}l, \tag{21}$$

where ${}^j C_L^k$ is a normalizing constant. Performing differentiation yields

$${}^j f_L^k({}^j l_k) = \frac{1}{{}^j C_L^k}\left[\hat{f}_j^k\left({}^j l_k + \frac{b_j}{2}\right)\int_{l=-\infty}^{{}^j l_k}\int_{r={}^j l_k}^{\infty} {}^{j-1}f_{LR}^k(l,r)\,\mathrm{d}r\,\mathrm{d}l + \int_{x={}^j l_k-b_j/2}^{{}^j l_k+b_j/2}\int_{r=x-b_j/2}^{\infty} \hat{f}_j^k(x)\, {}^{j-1}f_{LR}^k({}^j l_k, r)\,\mathrm{d}r\,\mathrm{d}x\right]. \tag{22}$$

Lemma. *The double integral over the joint density* ${}^{j-1}f_{LR}^k(l,r)$ *of the form*

$$\int_{l=-\infty}^{z}\int_{r=z}^{\infty} {}^{j-1}f_{LR}^k(l,r)\,\mathrm{d}r\,\mathrm{d}l \tag{23}$$

may be expressed in terms of the marginal densities as

$$\int_{y=-\infty}^{z} \{{}^{j-1}f_L^k(y) - {}^{j-1}f_R^k(y)\}\,\mathrm{d}y. \tag{24}$$

Proof. Expression (23) may be rewritten as

$$\int_{l=-\infty}^{z}\int_{r=-\infty}^{\infty} {}^{j-1}f_{LR}^k(l,r)\,\mathrm{d}r\,\mathrm{d}l - \int_{l=-\infty}^{z}\int_{r=-\infty}^{z} {}^{j-1}f_{LR}^k(l,r)\,\mathrm{d}r\,\mathrm{d}l. \tag{25}$$

Using the fact that ${}^{j-1}f_{LR}^k(l,r)$ is equal to zero for $l > r$, we may replace the upper limit of the first integral in the second expression by ∞. Interchanging the order of integration in the second expression yields

$$\int_{l=-\infty}^{z} {}^{j-1}f_L^k(l)\,\mathrm{d}l - \int_{r=-\infty}^{z} {}^{j-1}f_R^k(r)\,\mathrm{d}r, \tag{26}$$

which concludes the proof. □

Eq. (22) may be further simplified by using the lemma and by again using the fact that ${}^{j-1}f_{LR}^k(l,r)$ is equal to zero for $l > r$. The lower limit $x - b_j/2$ in the second integral of the second expression may be replaced by $-\infty$.

The desired recursion for the marginal density of the lower bound is now obtained as

$${}^j f_L^k(l) = \frac{1}{{}^j C_L^k}\left[\hat{f}_j^k\left(l + \frac{b_j}{2}\right)\int_{y=-\infty}^{l}\{{}^{j-1}f_L^k(y) - {}^{j-1}f_R^k(y)\}\,\mathrm{d}y + {}^{j-1}f_L^k(l)\int_{x=l-b_j/2}^{l+b_j/2}\hat{f}_j^k(x)\,\mathrm{d}x\right]. \tag{27}$$

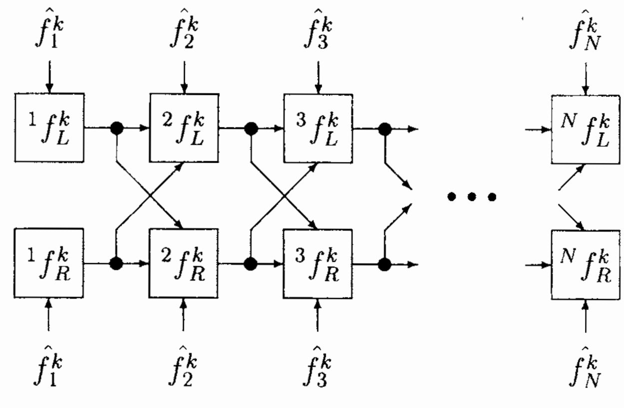

Fig. 3. Lattice-type recursion for the marginal densities of lower and upper bounds in the case of N information sources.

In analogy, the recursion for the marginal density of the upper bound may be derived as

$${}^j f_R^k(r) = \frac{1}{{}^j C_R^k}\left[\hat{f}_j^k\left(r - \frac{b_j}{2}\right)\int_{y=-\infty}^{r}\{{}^{j-1}f_L^k(y) - {}^{j-1}f_R^k(y)\}\,\mathrm{d}y + {}^{j-1}f_R^k(r)\int_{x=r-b_j/2}^{r+b_j/2}\hat{f}_j^k(x)\,\mathrm{d}x\right], \tag{28}$$

where ${}^j C_R^k$ is a normalizing constant.

This lattice-type recursion for the marginal densities of lower and upper bounds is depicted in Fig. 3 and initialized with

$${}^1f_L^k = \hat{f}_1^k\left(l + \frac{b_1}{2}\right), \qquad {}^1f_R^k = \hat{f}_1^k\left(r - \frac{b_1}{2}\right). \tag{29}$$

4.2. Gaussian noise densities

Again, $\hat{f}_j^k$ is a Gaussian density. The above expressions (27) and (28) for the marginal densities may be reduced to

$${}^j f_L^k(l) = \frac{1}{{}^j C_L^k}\left[\hat{f}_j^k\left(l + \frac{b_j}{2}\right)\int_{y=-\infty}^{l}\{{}^{j-1}f_L^k(y) - {}^{j-1}f_R^k(y)\}\,\mathrm{d}y + {}^{j-1}f_L^k(l)\left(\mathrm{erf}\left(\frac{l + b_j/2 - \hat{m}_j^k}{\hat{\sigma}_j^k}\right) - \mathrm{erf}\left(\frac{l - b_j/2 - \hat{m}_j^k}{\hat{\sigma}_j^k}\right)\right)\right], \tag{30}$$

$${}^j f_R^k(r) = \frac{1}{{}^j C_R^k}\left[\hat{f}_j^k\left(r - \frac{b_j}{2}\right)\int_{y=-\infty}^{r}\{{}^{j-1}f_L^k(y) - {}^{j-1}f_R^k(y)\}\,\mathrm{d}y + {}^{j-1}f_R^k(r)\left(\mathrm{erf}\left(\frac{r + b_j/2 - \hat{m}_j^k}{\hat{\sigma}_j^k}\right) - \mathrm{erf}\left(\frac{r - b_j/2 - \hat{m}_j^k}{\hat{\sigma}_j^k}\right)\right)\right]. \tag{31}$$

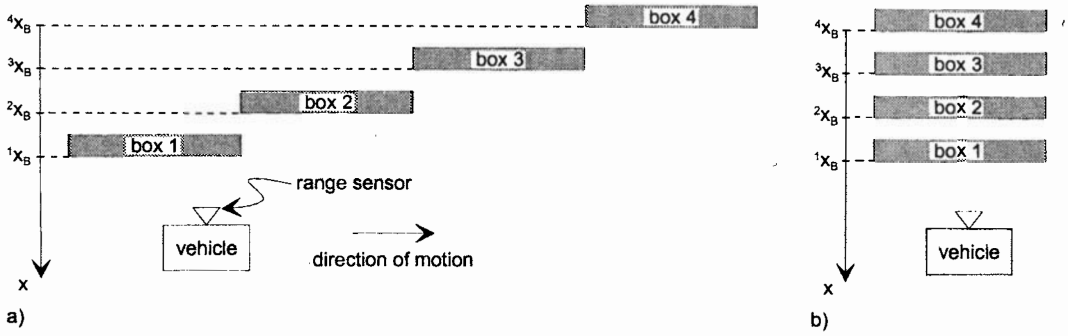

Fig. 4. Experimental setup for mobile robot localization: (a) sequential sampling, (b) simultaneous sampling.

5. Simulative verification

Since the authors' backgrounds are in mobile robot localization based on optical range data (Horn and Schmidt, 1995a, b), acoustical range data (Hanebeck and Schmidt, 1995), and angular measurements (Hanebeck and Schmidt, 1996), a simple scalar mobile robot localization problem is considered.

Two border cases of gathering information from several sources are illustrated by numerical examples: The first case consists of sampling the first source several times, then sampling the second source, and so on. The second case assumes that samples from all sources are available simultaneously.

Consider a vehicle equipped with a range sensor that measures the distance to a number of boxes as illustrated in Fig. 4. The box positions are known within a given geometric tolerance, i.e.,

$$^{i}x_B = {}^{i}\tilde{x}_B + {}^{i}\Delta x_B \quad \text{with } |{}^{i}\Delta x_B| \le \frac{b_i}{2}, \tag{32}$$

where ${}^{i}\tilde{x}_B$ denotes the unknown true value and ${}^{i}\Delta x_B$ is the unknown but bounded deviation of the nominal value ${}^{i}x_B$. The range sensor is corrupted by additive white Gaussian noise with zero mean and a variance σ_i^2 which depends on the surface characteristic of the box i. The measurement equation is thus given by

$$^{i}x_B + D_i^k = x + {}^{i}\Delta x_B + {}^{i}E_s^k, \tag{33}$$

where ${}^{i}E_s^k \sim N(0, \sigma_i)$, x denotes the vehicle position, and D_i^k is the measured distance. The two simulations are now performed for a true vehicle position $x = 200$ and the parameters given in Table 1.

$\hat{f}_i^k(x)$ is a Gaussian density where the mean and variance are recursively estimated by observing source i as (Schweppe, 1973)

$$\hat{m}_i^k = \frac{(\hat{\sigma}_i^{k-1})^{-2}\hat{m}_i^{k-1} + (\sigma_i)^{-2}({}^{i}x_B + d_i^k)}{(\hat{\sigma}_i^{k-1})^{-2} + (\sigma_i)^{-2}},$$
$$(\hat{\sigma}_i^k)^2 = ((\hat{\sigma}_i^{k-1})^{-2} + (\sigma_i)^{-2})^{-1}, \tag{34}$$

with $\hat{m}_i^0 = 0$, $(\hat{\sigma}_i^0)^{-1} = 0$.

Table 1
Parameters of localization experiment

Box	1	2	3	4
True value ${}^{i}\tilde{x}_B$	125	82	28	6
Nominal value ${}^{i}x_B$	120	80	40	0
Bound b_i	40	20	30	20
Standard deviation σ_i	10	10	10	10

The first simulation refers to Fig. 4a where the vehicle moves along the four boxes. The boxes are sampled sequentially with 100 samples for each box. We start with the first box, and simply obtain the marginal densities as the shifted versions of $\hat{f}_1^k(x)$ according to (29). These marginals serve as the initial densities for the recursion formulae (30) and (31) that are used for including information sources 2, 3, and 4 sequentially. The SSI recursion step from $i-1$ to i is performed whenever the Bayesian update (34) for information source i has been done, i.e., 100 times for boxes 2, 3, and 4, respectively. Fig. 5a depicts the response of the expected values of the lower and upper bounds of the set estimate. Sampling a specific box reduces the stochastic uncertainty. The initial deterministic uncertainty given by the interval [175, 215] is obviously reduced each time the box is changed: When changing from box 1 to box 2, the resulting intersection set without stochastic noise would be [188, 208], which is eventually approached when sampling box 2 for an infinite number of times. When traversing from box 2 to box 3, only the lower bound is updated, since the noise-free interval intersection would yield [197, 208]. Switching from box 3 to box 4 produces an update for the upper bound only, since the underlying interval intersection would yield [197, 204].

The second simulation refers to Fig. 4b where the vehicle samples all four boxes simultaneously at time k. The Bayesian update (34) is performed at every time k for each source i. Subsequently, the SSI recursions

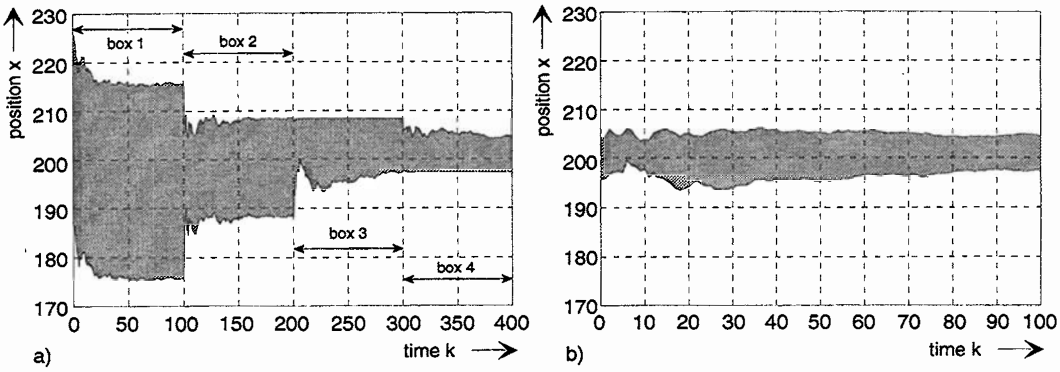

Fig. 5. Expected values of lower and upper bounds: (a) sequential sampling, (b) simultaneous sampling.

(30), (31) are performed, starting with Eq. (29), up to information source 4, at every time k. In this experiment, the underlying noise-free intersection set is [197, 204] for all k. This set is approached for an infinite number of measurements.

Fig. 5 clearly demonstrates the realistic quantification of the associated estimation uncertainty which is in sharp contrast to the optimism of point estimators. This feature may be exploited when attempting to navigate a mobile robot through narrow openings.

6. Conclusions

A combined Statistical and Set-theoretic Information (SSI) filter is introduced for the fusing of information from several sources which are simultaneously corrupted by a deterministic amplitude-bounded unknown bias error and a possibly unbounded random process. The new approach unites proven schemes for handling pure stochastic noise and for treating amplitude-bounded uncertainties. As a result, set estimates are provided rather than point estimates. Furthermore, the set bounds are uncertain in a statistical sense. Thus, these estimates do not suffer from the overoptimism encountered when one form of uncertainty is neglected. Monte Carlo simulations in the context of mobile robot localization demonstrate the effectiveness of the proposed approach. In addition, the simulation results demonstrate the two-fold uncertainty reduction during measurement acquisition.

The study presented only considered the scalar case. Nevertheless, generalization to higher dimensions is straightforward when attention is limited to hyper-rectangles parallel to the coordinate axes. The treatment of ellipsoidal set bounds is more involved, since ellipsoids are not closed under intersection, and the detection of ellipsoid overlap is tedious.

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References

Anderson, B. D. O., & Moore, J. B. (1979). *Optimal filtering*. Englewood Cliffs, NJ: Prentice-Hall.

Filipovic, V. Z. (1995). Identification in H_∞ in the presence of deterministic and stochastic noise. *Proc. 3rd European Control Conf.*, Rome, Italy (pp. 31–35).

Forsman, K., & Ljung, L. (1992). On the dead-zone in system identification. In C. Banyasz, L. Keviczky (Eds.), *Identification and system parameter estimation 1991*. Selected Papers from the *9th IFAC/IFORS Symp. on System Identification* (pp. 959–963). Budapest, Hungary, July 1991. Oxford, UK: Pergamon Press.

Friedland, B. (1969). Treatment of bias in recursive filtering. *IEEE Trans. Automat. Control*, *AC-14*(4), pp. 359–367.

Hanebeck, U. D., & Schmidt, G. (1995). A new high performance multisonar system for fast mobile robot applications. In V. Graefe (Ed.), *Intelligent robots and systems 1994 (IROS '94)* (pp. 1–14). Amsterdam: Elsevier Science.

Hanebeck, U. D., & Schmidt, G. (1996). Set-theoretic localization of fast mobile robots using an angle measurement technique. *Proc. 1996 IEEE Int. Conf. on Robotics and Automation*, Minneapolis, MN (pp. 1387–1394).

Hochwald, B., & Nehorai, A. (1995). Minimum bias priors for estimating parameters of additive terms in state-space models. *IEEE Trans. Automat. Control AC-40*(4), pp. 684–693.

Horn, J., & Schmidt, G. (1995a). Continuous localization for long-range indoor navigation of mobile robots. *Proc. 1995 IEEE Int. Conf. on Robotics and Automation*, Nagoya, Japan (pp. 387–394).

Horn, J., & Schmidt, G. (1995b). Continuous localization of a mobile robot based on 3D-laser-range-data, predicted sensor images, and dead-reckoning. *Robotics and Autonomous Systems* (Vol. 14, nos. 2–3, pp. 99–118). Amsterdam: Elsevier Science.

Papoulis, A. (1984). *Probability, random variables, and stochastic processes* (2nd ed.). New York: McGraw-Hill. 1984.

Schweppe, F.C. (1973). *Uncertain dynamic systems*. Englewood Cliffs, NJ: Prentice-Hall.

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