New Estimators for Mixed Stochastic and Set Theoretic Uncertainty Models: The Vector Case

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Keywords: Estimation theory, filtering techniques, mixed noise models, bounded noise, stochastic noise.

Abstract

This work presents new results for state estimation based on noisy observations suffering from two different types of uncertainties. The first uncertainty is a stochastic process with given statistics. The second uncertainty is only known to be bounded, the exact underlying statistics are unknown. State estimation tasks of this kind typically arise in target localization, navigation, and sensor data fusion. A new estimator has been developed, that combines set theoretic and stochastic estimation in a rigorous manner. The estimator is efficient and, hence, well-suited for practical applications. It provides a continuous transition between the two classical estimation concepts, because it converges to a set theoretic estimator, when the stochastic error goes to zero, and to a Kalman filter, when the bounded error vanishes. In the mixed noise case, the new estimator provides solution sets that are uncertain in a statistical sense.

1 Introduction

Often, the state of a dynamic system has to be estimated on the basis of uncertain measurements. Applications include vehicle or missile localization, target tracking, navigation, and sensor data fusion. This paper focuses on vector measurements of the desired state for the special case, that the dimension of the measurement vector is equal to the dimension of the state vector. The resulting estimation procedure is then used for recursively incorporating vector measurements into a state estimate. On the other hand, the estimation procedure can also be used for the combination of individual estimates of the desired state, for example provided by several different estimators.

In general, the goal of an estimation procedure is to reduce the uncertainty about the system's state as much as possible. When an appropriate system model together with noise statistics is given, the Kalman filter and its descendants [1] have been successfully applied for more than 30 years. However, in the applications cited above, a detailed statistical noise model is often either not available or impractical. Special caution is in order when neglecting strongly correlated noise or systematic errors. In that case, Kalman filter estimates tend to be overoptimistic [10], i.e., the covariance estimate becomes unrealistically small. Several heuristics have been suggested for coping with this problem, ranging from artificially increasing the covariance from time to time to employing nonlinear pre–filters. Of course, these techniques do not provide optimal estimators.

In some situations, although a statistical noise description cannot be given, bounds for the noise can be provided. This may be the case for unmodeled dynamics, unmodeled nonlinearities, correlated noise, and systematic errors. In that case, set theoretic estimation can be applied [12], which often leads to good results [4]. However, when additional uncorrelated noise is present, the error bounds become unnecessarily conservative.

In [5, 8], a concept for estimation in the presence of both bounded and stochastic uncertainties has been introduced. The proposed algorithm for the case of a scalar state is exact, but computationally complex. In [6, 7], an approximate solution for the case of a scalar state has been derived, that is computationally attractive.

This paper presents a unification of set theoretic and stochastic estimation for arbitrary dimensional states and measurements, when state and measurement vector are of the same dimension. The new estimator bridges the gap between both estimation schemes, because it becomes a set theoretic estimator, when the stochastic error goes to zero, and it becomes a Kalman filter, when the bounded error vanishes. When both types of uncertainty are present, the new estimator provides solution sets that are uncertain in a statistical sense.

In Sec. 2, a formulation of the estimation problem with a mixed stochastic and set theoretic uncertainty model is given. In Sec. 3, the estimation concept is presented. In Sec. 4, the estimation problem is solved on the basis of a sum approximation. Section 5 then presents a simulative example that demonstrates the performance of the new estimator.

2 Problem Formulation

We consider two uncertain observations of an unknown state $\underline{z}$ given by

$$\begin{aligned} \hat{\underline{x}} &= \underline{z} + \underline{e}_x + \underline{c}_x , \\ \hat{\underline{y}} &= \underline{z} + \underline{e}_y + \underline{c}_y \ . \end{aligned} \tag{1}$$

$\hat{\underline{x}}$ and $\hat{\underline{y}}$ suffer from two types of additive noise [2, 3]: 1) Uncertainties $\underline{e}_x$, $\underline{e}_y$ bounded by the sets

$$\begin{aligned} \mathcal{E}_x &= \left\{ \underline{e}_x : \underline{e}_x^T \mathbf{E}_x^{-1} \underline{e}_x \leq 1 \right\} \\ \mathcal{E}_y &= \left\{ \underline{e}_y : \underline{e}_y^T \mathbf{E}_y^{-1} \underline{e}_y \leq 1 \right\} \end{aligned} \ ,$$

where the only prior knowledge is their boundedness and 2) Gaussian noise sources $\underline{c}_x$, $\underline{c}_y$, with $\underline{c}_x \sim N(\underline{0}, \mathbf{C}_x)$, $\underline{c}_y \sim N(\underline{0}, \mathbf{C}_y)$, which are assumed to be uncorrelated.

3 The Estimation Concept

First, assume that $\hat{\underline{x}}$, $\hat{\underline{y}}$ can be observed without stochastic uncertainty. Then, since there is no prior information about $\underline{e}_x$, $\underline{e}_y$ besides their boundedness, we make the worst case assumption that $\underline{e}_x$, $\underline{e}_y$ are fully correlated. In that case, a set theoretic estimator is appropriate for fusing the information sources. An efficient form of a set theoretic estimator, which is based on the convex combination of the original sets, is given by the ellipsoidal set [12]

$$\mathcal{Z} = \{ \underline{z} : (\underline{z} - \hat{\underline{z}})^T \mathbf{E}_z^{-1} (\underline{z} - \hat{\underline{z}}) \leq 1 \} \ .$$

The midpoint of the ellipsoid is given by

$$\hat{\underline{z}} = \mathbf{W}_x \, \hat{\underline{x}} + \mathbf{W}_y \, \hat{\underline{y}}$$

with weighting factors

$$\begin{aligned} \mathbf{W}_x &= (0.5 - \lambda) \mathbf{P}_z \, \mathbf{E}_x^{-1} \ , \\ \mathbf{W}_y &= (0.5 + \lambda) \mathbf{P}_z \, \mathbf{E}_y^{-1} \ , \end{aligned}$$

where $\mathbf{W}_x + \mathbf{W}_y = \mathbf{I}$. The appropriate selection of the parameter $\lambda \in [-0.5, 0.5]$ will be discussed later. The set theoretic uncertainty is given by

$$\mathbf{E}_z = d \, \mathbf{P}_z$$

with

$$\begin{aligned} d &= 1 - \left(0.25 - \lambda^2\right) (\hat{\underline{x}} - \hat{\underline{y}})^T \mathbf{E}_y^{-1} \, \mathbf{P}_z \, \mathbf{E}_x^{-1} (\hat{\underline{x}} - \hat{\underline{y}}) \ , \\ \mathbf{P}_z &= \left\{ (0.5 - \lambda) \mathbf{E}_x^{-1} + (0.5 + \lambda) \mathbf{E}_y^{-1} \right\}^{-1} \ . \end{aligned}$$

It is important to note that the set theoretic uncertainty $\mathbf{E}_z$ depends on the actual observations $\hat{\underline{x}}$, $\hat{\underline{y}}$. From now on, we will use the outer bound of $\mathbf{E}_z$ given by setting $d = 1$. The so obtained ellipsoid always contains the true set. Most importantly, $\mathbf{E}_z$ does not depend on the actual observations, which simplifies some of the following derivations. Hence, we obtain the ellipsoid

$$\mathcal{Z} = \{ \underline{z} : (\underline{z} - \hat{\underline{z}})^T \mathbf{E}_z^{-1} (\underline{z} - \hat{\underline{z}}) \leq 1 \} \ ,$$

with midpoint

$$\hat{\underline{z}} = \mathbf{W}_x \, \hat{\underline{x}} + \mathbf{W}_y \, \hat{\underline{y}} \tag{2}$$

and weighting factors

$$\begin{aligned} \mathbf{W}_x &= (0.5 - \lambda) \mathbf{E}_z \, \mathbf{E}_x^{-1} \ , \\ \mathbf{W}_y &= (0.5 + \lambda) \mathbf{E}_z \, \mathbf{E}_y^{-1} \ , \end{aligned} \tag{3}$$

for $\lambda \in [-0.5, 0.5]$. The set theoretic uncertainty is given by

$$\mathbf{E}_z = \left\{ (0.5 - \lambda) \mathbf{E}_x^{-1} + (0.5 + \lambda) \mathbf{E}_y^{-1} \right\}^{-1} \ . \tag{4}$$

However, $\hat{\underline{x}}$, $\hat{\underline{y}}$ cannot be observed directly, but are corrupted by Gaussian noise. Hence, $\hat{\underline{z}}$ is a random variable with statistics that can be obtained from (2). Since $\mathbf{E}_z$ in (4) does not depend on the actual observations, it is not a random variable.

The remainder of this paper is concerned with calculating the density of $\hat{\underline{z}}$. Since the exact density is not useful for practical applications, it is approximated by a weighted sum of Gaussians. This approximate density approaches the exact density for an infinite number of terms. A finite approximation is useful for data–recursive estimation.

The total uncertainty of the estimate is then given by the set theoretic uncertainty in (4) and the stochastic uncertainty described by the density of $\hat{\underline{z}}$.

4 Approximate Solution for the Density

For nonsingular $\mathbf{W}_x$, the density $f_z(\hat{\underline{z}})$ of $\hat{\underline{z}}$ is given by

$$f_z(\hat{\underline{z}}) = \frac{1}{|\mathbf{W}_x|} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{xy} \left(\mathbf{W}_x^{-1} (\hat{\underline{z}} - \mathbf{W}_y \hat{\underline{y}}), \hat{\underline{y}} \right) \, d\hat{\underline{y}} \ .$$

From (1) we deduce, that $\hat{\underline{x}}$ and $\hat{\underline{y}}$ are not noisy estimates of $\underline{z}$, but of $\underline{z}+\underline{e}_x$ and $\underline{z}+\underline{e}_y$, respectively. Hence, we bound the difference $\hat{\underline{x}} - \hat{\underline{y}}$ by the Minkowski sum of $\mathbf{E}_x$ and $\mathbf{E}_y$, which is not an ellipsoidal set. However, the Minkowski sum can easily be bounded by an ellipsoid according to [12]

$$(\underline{\hat{x}} - \underline{\hat{y}})^T \mathbf{B}^{-1} (\underline{\hat{x}} - \underline{\hat{y}}) \leq 1 \ ,$$

where $\mathbf{B}$ is given by

$$\mathbf{B} = \frac{1}{0.5 - \kappa} \mathbf{E}_x + \frac{1}{0.5 + \kappa} \mathbf{E}_y \ ,$$

for $\kappa \in (-0.5, 0.5)$. κ is selected in such a way, that the volume of the resulting bounding ellipsoid for the exact Minkowski sum is minimized. This leads to

$$f_{xy}(\underline{\hat{x}}, \underline{\hat{y}}) = \begin{cases} c_{xy}\, f_x(\underline{\hat{x}})\, f_y(\underline{\hat{y}}) & \text{for } (\underline{\hat{x}} - \underline{\hat{y}})^T \mathbf{B}^{-1} (\underline{\hat{x}} - \underline{\hat{y}}) \leq 1 \\ 0 & \text{elsewhere} \end{cases} \quad (5)$$

with normalizing constant c_{xy}. Defining an indicator function

$$I(\underline{\hat{x}}, \underline{\hat{y}}) = \begin{cases} 1 & \text{for } (\underline{\hat{x}} - \underline{\hat{y}})^T \mathbf{B}^{-1} (\underline{\hat{x}} - \underline{\hat{y}}) \leq 1 \\ 0 & \text{elsewhere} \end{cases} \ ,$$

(5) simplifies to

$$f_{xy}(\underline{\hat{x}}, \underline{\hat{y}}) = c_{xy}\, f_x(\underline{\hat{x}})\, f_y(\underline{\hat{y}})\, I(\underline{\hat{x}}, \underline{\hat{y}}) \ .$$

The key idea to finding an approximate solution for the probability density function is to approximate the indicator function by a weighted sum of Gaussians according to

$$I(\underline{\hat{x}}, \underline{\hat{y}}) \approx \sum_{i=1}^{N} \exp\left\{ -\frac{1}{2} \left(\underline{\hat{x}} - \underline{\hat{y}} - \underline{m}_{g,i}\right)^T \mathbf{C}_{g,i}^{-1} \left(\underline{\hat{x}} - \underline{\hat{y}} - \underline{m}_{g,i}\right) \right\} \ ,$$

with $\underline{m}_{g,i}$ and symmetric, positive definite matrices $\mathbf{C}_{g,i}$ appropriately chosen. After a lot of manipulation, the approximate density of $\underline{\hat{z}}$ can be written as

$$f_z\left(\underline{\hat{z}}\right) \approx c \sum_{i=1}^{N} g_i \exp\left\{ -\frac{1}{2} \left(\underline{\hat{z}} - \underline{\bar{z}}_i\right)^T \mathbf{C}_{z,i}^{-1} \left(\underline{\hat{z}} - \underline{\bar{z}}_i\right) \right\}$$

with normalizing constant c, weighting terms

$$g_i = \exp\left\{ -\frac{1}{2} \left(\underline{\hat{x}} - \underline{\hat{y}} - \underline{m}_{g,i}\right)^T \left(\mathbf{C}_x + \mathbf{C}_y + \mathbf{C}_{g,i}\right)^{-1} \left(\underline{\hat{x}} - \underline{\hat{y}} - \underline{m}_{g,i}\right) \right\} \ , \quad (6)$$

the individual means

$$\begin{aligned} \underline{\bar{z}}_i = \ & \left(\mathbf{C}_y + \mathbf{W}_x \mathbf{C}_{g,i}\right) \left(\mathbf{C}_x + \mathbf{C}_y + \mathbf{C}_{g,i}\right)^{-1} \underline{\hat{x}} \\ & + \left(\mathbf{C}_x + \mathbf{W}_y \mathbf{C}_{g,i}\right) \left(\mathbf{C}_x + \mathbf{C}_y + \mathbf{C}_{g,i}\right)^{-1} \underline{\hat{y}} \\ & + \left(\mathbf{W}_x \mathbf{C}_x - \mathbf{W}_y \mathbf{C}_y\right) \left(\mathbf{C}_x + \mathbf{C}_y + \mathbf{C}_{g,i}\right)^{-1} \underline{m}_{g,i} \end{aligned} \quad (7)$$

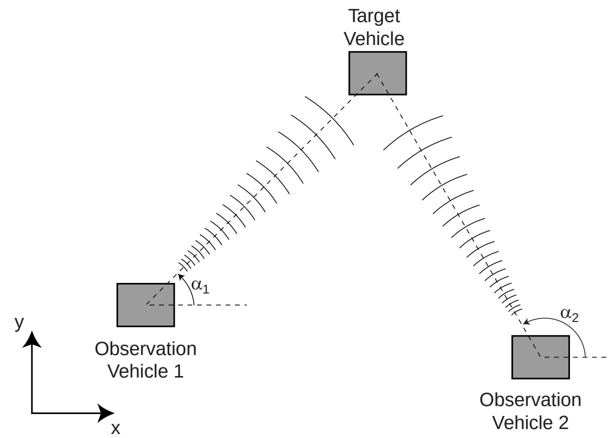

Figure 1: Setup for simulative verification of the proposed estimator: Estimating the position of a target vehicle with two observation vehicles.

and

$$\begin{aligned} \mathbf{C}_{z,i} = \ & \mathbf{W}_x \mathbf{C}_x \left(\mathbf{C}_x + \mathbf{C}_y + \mathbf{C}_{g,i}\right)^{-1} \mathbf{C}_y \\ & + \mathbf{W}_y \mathbf{C}_y \left(\mathbf{C}_x + \mathbf{C}_y + \mathbf{C}_{g,i}\right)^{-1} \mathbf{C}_x \\ & + \mathbf{W}_x \mathbf{C}_x \left(\mathbf{C}_x + \mathbf{C}_y + \mathbf{C}_{g,i}\right)^{-1} \mathbf{C}_{g,i} \mathbf{W}_x^T \\ & + \mathbf{W}_y \mathbf{C}_y \left(\mathbf{C}_x + \mathbf{C}_y + \mathbf{C}_{g,i}\right)^{-1} \mathbf{C}_{g,i} \mathbf{W}_y^T \ . \end{aligned} \quad (8)$$

Using this approximate probability density function, we can easily calculate the approximate moments of $\underline{\hat{z}}$ up to second order according to

$$E[\underline{\hat{z}}] \approx \frac{\sum_{i=1}^{N} g_i \underline{\bar{z}}_i}{\sum_{i=1}^{N} g_i} \quad (9)$$

and

$$E\left[\underline{\hat{z}}^2\right] \approx \frac{\sum_{i=1}^{N} g_i \left(C_{z,i} + \underline{\bar{z}}_i \underline{\bar{z}}_i^T\right)}{\sum_{i=1}^{N} g_i} \ .$$

The covariance matrix of $\underline{\hat{z}}$ is given by

$$\mathbf{C}_z = E\left[\underline{\hat{z}}^2\right] - E[\underline{\hat{z}}] E[\underline{\hat{z}}]^T \ . \quad (10)$$

5 Simulative Results

For demonstrating the performance of the proposed estimator, we consider the problem of estimating the position of a target vehicle with two observation vehicles, Fig. 1.

The two observation vehicles are equipped with radar sensors for measuring range and bearing. These sensors are used for estimating the relative position of the target vehicle with respect to the observation vehicles. Furthermore, the observation vehicles can determine their ego–positions, which are used for transforming the relative position estimates to absolute position estimates with respect to an inertial coordinate system.

The radar measurements are subject to random noise in both range and bearing. The noise sources are assumed to be white, Gaussian random processes with zero mean. For the setup depicted in Fig. 1, the resulting noises in the relative position estimates are two–dimensional random processes with the following statistics. For the first observation vehicle, the standard deviation is assumed to be $\sigma_{1,r} = 1.0$ in the range measurement, and $\sigma_{1,o} = 4.0$ orthogonal to the range measurement. Hence, the measurement covariance matrix is given by

$$\mathbf{C}_1 = \mathbf{T}_1 \begin{bmatrix} \sigma_{1,r}^2 & 0 \\ 0 & \sigma_{1,o}^2 \end{bmatrix} \mathbf{T}_1^T$$

with the rotation matrix

$$\mathbf{T}_1 = \begin{bmatrix} \cos(\alpha_1) & -\sin(\alpha_1) \\ \sin(\alpha_1) & \cos(\alpha_1) \end{bmatrix} \; , \quad \alpha_1 = 45^o \; .$$

For the second observation vehicle, we have $\sigma_{2,r} = 1.0$, $\sigma_{2,o} = 4.0$ and hence the covariance matrix is given by

$$\mathbf{C}_2 = \mathbf{T}_2 \begin{bmatrix} \sigma_{2,r}^2 & 0 \\ 0 & \sigma_{2,o}^2 \end{bmatrix} \mathbf{T}_2^T$$

with the rotation matrix

$$\mathbf{T}_2 = \begin{bmatrix} \cos(\alpha_2) & -\sin(\alpha_2) \\ \sin(\alpha_2) & \cos(\alpha_2) \end{bmatrix} \; , \quad \alpha_2 = 120^o \; .$$

The nominal positions of the observation vehicles used by the filter are given by

$$\underline{m}_1 = \begin{bmatrix} 150 \\ 150 \end{bmatrix} \; ,$$
$$\underline{m}_2 = \begin{bmatrix} 753.55 \\ 70.54 \end{bmatrix} \; .$$

The true positions of the observation vehicles are given by

$$\tilde{\underline{m}}_1 = \begin{bmatrix} 150 - 0.3 \\ 150 - 0.5 \end{bmatrix} \; ,$$
$$\tilde{\underline{m}}_2 = \begin{bmatrix} 753.55 - 0.3 \\ 70.54 - 0.5 \end{bmatrix} \; ,$$

which are unknown to the filter. The bounds for the uncertainties in the positions of the observation vehicles are assumed to be ellipsoids defined by

$$\mathbf{E}_1 = \begin{bmatrix} 1.0^2 & -(0.4^2) \\ -(0.4^2) & 0.8^2 \end{bmatrix} \; ,$$
$$\mathbf{E}_2 = \begin{bmatrix} 0.6^2 & 0.2^2 \\ 0.2^2 & 1.4^2 \end{bmatrix} \; .$$

The best possible estimate even without random noise would be the set given by intersecting these two ellipsoids. The true position of the target vehicle is given by

$$\tilde{\underline{m}}_T = \begin{bmatrix} 503.55 \\ 503.55 \end{bmatrix} \; ,$$

and is unknown to the filter. At each time step k, $k = 1, \dots, 10000$, one of the observation vehicles measures the position of the target vehicle. For k odd, observation vehicle 1 performs a measurement, for k even, observation vehicle 2 performs a measurement. The proposed estimator recursively incorporates the individual position estimates from the observation vehicles. This is performed by first calculating the set of mean values according to (7), the set of covariance matrices according to (8), and the set of weights according to (6). Subsequently, the mean value $E[\hat{\underline{m}}_T]$ and the covariance matrix $\mathbf{C}_T$ of the position of the target vehicle are calculated according to (9), (10). In addition, the set theoretic part of the total uncertainty is calculated according to (4). At each recursion step, the parameter λ in (3) is chosen such that

$$\det(\mathbf{E}_T + \mathbf{C}_T)$$

is minimized. The evolution of the resulting confidence set is depicted in Fig. 3. The confidence set has been calculated as the Minkowski sum of $\mathbf{E}_T$ and 9 times $\mathbf{C}_T$ centered at $E[\hat{\underline{m}}_T]$. **Note:** The confidence set for $k \to \infty$ bounds the exact set (the intersection of the set theoretic uncertainties due to the uncertain positions of the observation vehicles) from above, and hence contains the true state.

To compare these results with standard Kalman filtering, the position uncertainties of the observation vehicles are treated as additional independent white Gaussian noise with zero mean and variance $\mathbf{E}_1$, $\mathbf{E}_2$, respectively. This results in a total measurement covariance of $\mathbf{C}_1 + \mathbf{E}_1$, $\mathbf{C}_2 + \mathbf{E}_2$, respectively. The evolution of the resulting confidence set is depicted in Fig. 3. The confidence set has been calculated based on 9 times the Kalman filter covariance matrix centered at $\hat{\underline{m}}_T^{\text{Kalman}}$. **Note:** The confidence set for $k \to \infty$ does *not* contain the true position of the target vehicle.

6 Conclusions

Many estimation problems can be approached as a mixed noise problem, i.e., the arising uncertainties can be modeled as being additively composed of both 1) noise with known bounds and 2) noise with known statistics.

A new estimator for fusing data subject to mixed noise has been derived for arbitrary dimensional states and measurements, when state and measurement vector are of the same dimension. The proposed estimator unifies stochastic and set theoretic filtering. It converges to a stochastic estimator, when only noise with known statistics is

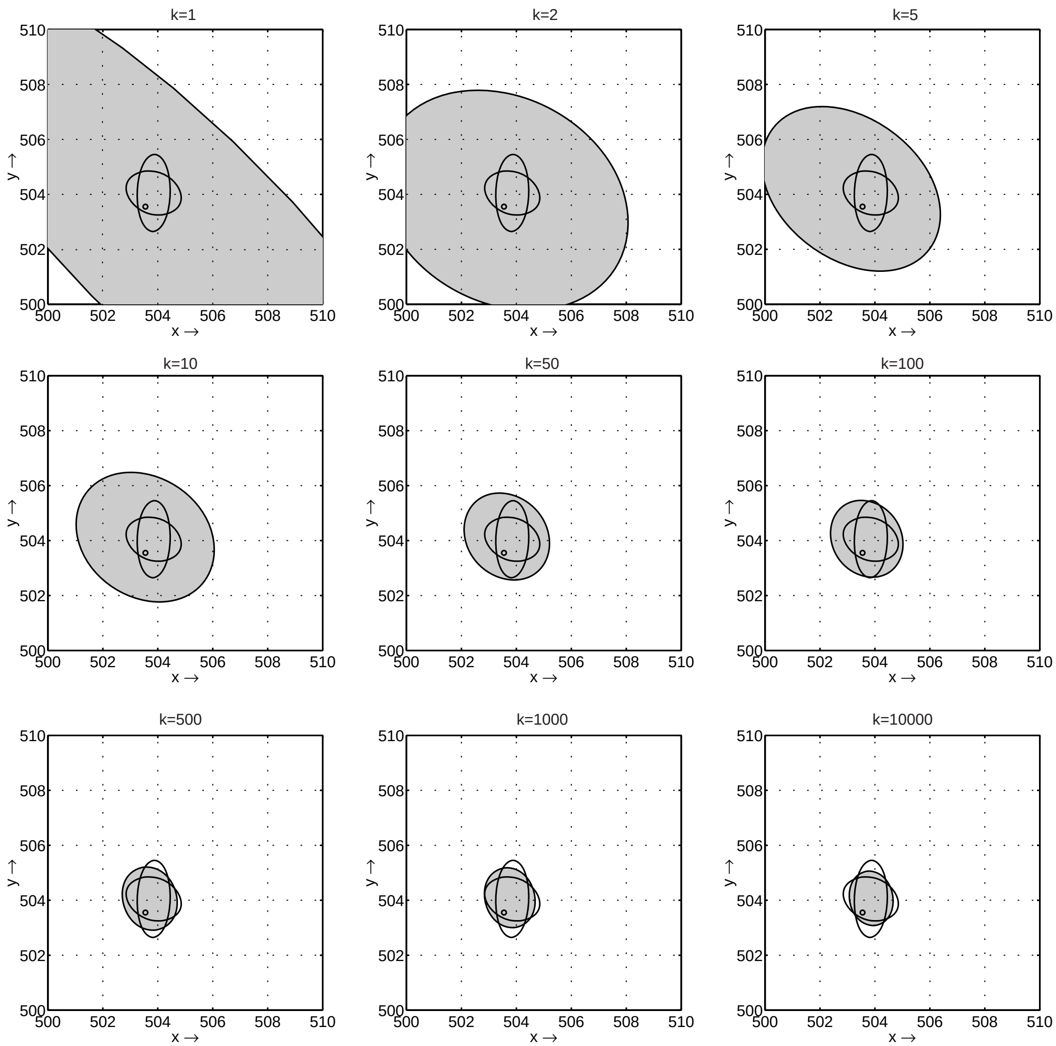

Figure 2: Results of applying the new estimator: Evolution of confidence set over time. The true position is marked by a small circle. Additionaly, the set theoretic uncertainties due to the uncertain positions of the observation vehicles are shown.

present. On the other hand, it converges to a set theoretic estimator, when only noise with known bounds is present. When both types of uncertainty are present simultaneously, the new estimator provides estimates comprising a mean value, an ellipsoidal bound for the set theoretic uncertainty, and a weighted sum of Gaussian densities quantifying the stochastic uncertainty.

The proposed estimator is efficient and, hence, well–suited for practical applications.

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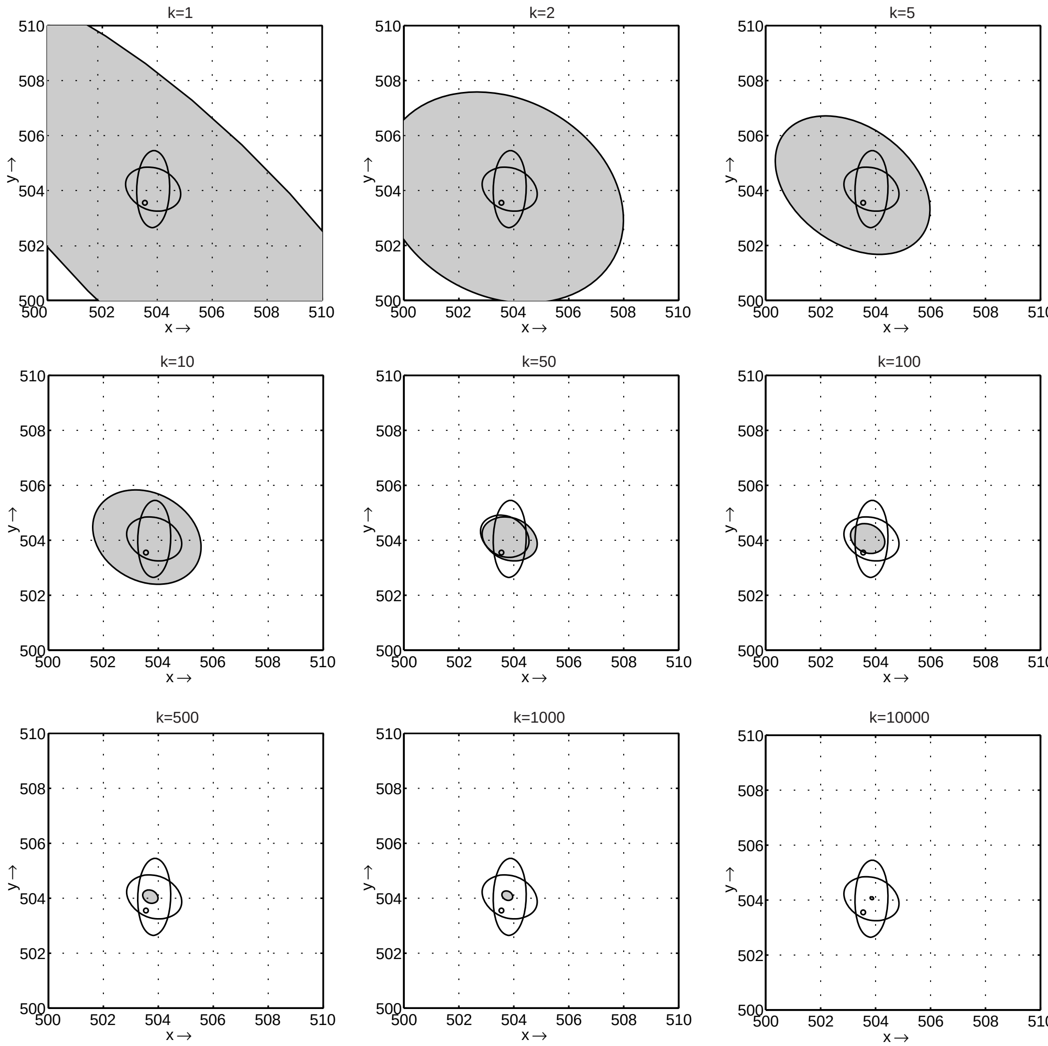

Figure 3: Results of applying the Kalman filter: Evolution of confidence set over time. The true position is marked by a small circle. Additionaly, the set theoretic uncertainties due to the uncertain positions of the observation vehicles are shown.

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