Optimal Gaussian Filtering for Polynomial Systems Applied to Association-free Multi-Target Tracking

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***Abstract*—This paper is about tracking multiple targets with the so-called Symmetric Measurement Equation (SME) filter. The SME filter uses symmetric functions, e.g., symmetric polynomials, in order to remove the data association uncertainty from the measurement equation. By this means, the data association problem is converted to a nonlinear state estimation problem. In this work, an efficient optimal Gaussian filter based on analytic moment calculation for discrete-time multi-dimensional polynomial systems corrupted with Gaussian noise is derived, and then applied to the polynomial system resulting from the SME filter. The performance of the new method is compared to an UKF implementation by means of typical multiple target tracking scenarios.**

Keywords: Gaussian filtering, polynomial systems, SME filter, multi-target tracking.

I. Introduction

Tracking multiple targets based on noisy measurements is a frequently occuring problem in many applications such as surveillance [1], [2]. A major problem in multi-target tracking is that the measurement-to-target association is unknown, i.e., the target from which a measurement originates is not given. Many different solutions and methodologies for dealing with this data association uncertainty can be found in literature [1]–[3]. For instance, the Multi-Hypothesis-Tracker (MHT) [4] maintains all feasible data association hypotheses over a finite time horizon. The well-known Joint Probabilistic Data Association Filter (JPDAF) [3] combines all single target estimates according to association probabilities.

Association-free methods such as the PHD-filter [5], [6] do not explicitly evaluate association hypotheses. This paper is about an association-free method called Symmetric Measurement Equation (SME) filter [7]–[10]. The SME filter uses symmetric functions in order to remove the data association uncertainty from the measurement equation. A function is called symmetric if a permutation of its arguments does not change the result. Usually, the SME filter is based on symmetric polynomials, and hence, the resulting measurement equation is a polynomial as well. As a consequence, the data association problem is converted to a nonlinear estimation problem with a polynomial measurement equation.

In this work, we aim at using a Gaussian state estimator for the arising polynomial equations of the SME filter. Examples for a Gaussian state estimator are the well-known Extended Kalman filter (EKF) [11] or deterministic sampling approaches such as the UKF [12] or [13]. The Divided Difference Filter (DDF) performs a derivative-free approximation of the system functions based on Stirling's formula [14], [15]. The Polynomial Extended Kalman filter [16] employs a Carleman linearization in order to obtain a bilinear system.

A well-known and widely-used techniques is the analytic moment calculation of nonlinear transformed random variables. For instance, the Divided Difference Filter (DDF) [14] performs analytic moment calculation for a second-order polynomial approximation of the system functions. In [17]–[19], the moments of a second-order polynomial system function are directly be computed in closed form und used for optimal estimation. Recently, analytic methods have been combined with approximate methods [20]–[23]. In [24], an optimal Kalman filter for one-dimensional polynomial systems is derived and in [25], a quasi-Gaussian Kalman filter for continuous dynamic systems is presented. Analytic moment calculation has been used in a variety of applications such as localization [17] or tracking [18], [19] and proven to be a promising alternative to approximate solutions as it gives the optimal solution in closed form and no parameter tuning is required. To the best of our knowledge, analytic moment calculation has not been applied to the SME filter yet.

The contributions of this work are the followings. First, we describe an efficient black-box Gaussian-assumed Bayesian filter for multi-dimensional polynomial systems based on analytic moment calculation (see Section II). For this purpose, an automatic efficient method for computing the first two moments of polynomially transformed Gaussian random variables is derived based on a formula for the expectation of products of Gaussian random variables introduced in [26]. The filter is then applied for tracking multiple targets with the *SME* approach [7], [9], [10] using symmetric polynomial functions. Simulation results show that the introduced filter is feasible even for high-dimensional polynomial systems and outperforms an UKF implementation in typical tracking scenarios (see Section III).

II. Gaussian Assumed Bayesian Filter for Polynomial Systems

In this work, we consider a discrete time nonlinear dynamic system of the form

$$\underline{\boldsymbol{x}}_{k+1} = a_k(\underline{\boldsymbol{x}}_k, \underline{u}_k, \underline{\boldsymbol{w}}_k) \ , \tag{1}$$

with state vector $\underline{\boldsymbol{x}}_k$, system function $a_k(\cdot,\cdot)$, system input $\underline{u}_k$, and white Gaussian system noise $\underline{\boldsymbol{w}}_k$. The state vector $\underline{\boldsymbol{x}}_k$ is related to the measurements according to a measurement equation of the form

$$\underline{\boldsymbol{y}}_k = h_k(\underline{\boldsymbol{x}}_k, \underline{\boldsymbol{v}}_k) \ , \tag{2}$$

where $h_k(\cdot,\cdot)$ is the measurement function and $\underline{\boldsymbol{v}}_k$ is white Gaussian measurement noise. In the following, we focus on polynomial functions $a_k(\cdot,\cdot)$ and $h_k(\cdot,\cdot)$. The goal is to estimate the hidden state $\underline{\boldsymbol{x}}_k$ based on observations of $\underline{\boldsymbol{y}}_k$.

A. Gaussian-assumed Bayesian Filter

In the following, the well-known scheme for deriving a Bayesian state estimator based on Gaussian densities is described. A Bayesian estimator for the state $\underline{\boldsymbol{x}}_k$ [11]–[13], [17], [24] recursively maintains a probability density function for the random vector $\underline{\boldsymbol{x}}_k$ by alternating a time update and a measurement update.

Given the current probability density function $f^e(\underline{x}_{k-1})$ at time step $k-1$, the time update step computes the prediction $f^p(\underline{x}_k)$ for time step k according to the Chapman-Kolomogorov equation

$$f^p(\underline{x}_k) = \int f(\underline{x}_k|\underline{x}_{k-1}) \cdot f^e(\underline{x}_{k-1}) d\underline{x}_{k-1} \ . \tag{3}$$

In the measurement update step, the prediction $f^p(\underline{x}_k)$ is updated with the measurement $\hat{\underline{y}}_k$ according to Bayes' rule

$$f^e(\underline{x}_k) = \alpha_k \cdot f^L(\hat{\underline{y}}_k|\underline{x}_k) \cdot f^p(\underline{x}_k) \ , \tag{4}$$

where $f^L(\hat{\underline{y}}_k|\underline{x}_k)$ is the measurement likelihood function and α_k is a normalization factor.

A Gaussian-assumed Bayesian estimator approximates all densities with a Gaussian density, i.e., $f^p(\underline{x}_k) \approx \mathcal{N}(\underline{x}_k - \underline{\mu}_k^x, \Sigma_k^x)$ and $f^e(\underline{x}_k) \approx \mathcal{N}(\underline{x}_k - \underline{\mu}_k^x, \Sigma_k^x)$.

The key technique [11]–[13], [17], [24] used here for performing the measurement and time update based on Gaussian densities is the computation of the first two moments of the nonlinear transformation of a Gaussian random vector. For a given Gaussian random vector $\mathcal{N}(\underline{z} - \underline{\mu}^z, \Sigma)$, the probability density of $g(\underline{\boldsymbol{z}})$, where $g(\cdot)$ is a polynomial transformation, can be *approximated* with a Gaussian density

$$g(\underline{\boldsymbol{z}}) \sim \mathcal{N}(\underline{\mu}^g, \Sigma^g) \ , \tag{5}$$

where $\underline{\mu}^g$ and Σ^g are the mean and covariance matrix of $g(\underline{\boldsymbol{z}})$.

Usually, the moments $\underline{\mu}^g$ and Σ^g are approximated. For instance, sample-based approaches such as [12] or [13] propagate samples through the nonlinear mapping in order get approximations of the moments.

Based on this technique, the time update (3) can directly be performed as it is a stochastic forward mapping from $\underline{\boldsymbol{x}}_k$ to $\underline{\boldsymbol{x}}_{k+1}$.

In order to perform the measurement update (4), the predicted measurement is computed by approximating the probability distribution of the random vector

$$\begin{bmatrix} \underline{\boldsymbol{x}}_k \\ \underline{\boldsymbol{y}}_k \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{x}}_k \\ h_k(\underline{\boldsymbol{x}}_k, \underline{\boldsymbol{v}}_k) \end{bmatrix}$$

with a Gaussian distribution with mean $\begin{bmatrix} \underline{\mu}^x \\ \underline{\mu}^y \end{bmatrix}$ and covariance matrix $\begin{bmatrix} \Sigma^x & \Sigma^{yx} \\ \Sigma^{xy} & \Sigma^y \end{bmatrix}$. The updated estimate and covariance matrix then result from conditioning this joint density on $\hat{\underline{y}}_k$ based on the Kalman filter equations

$$\begin{aligned} \underline{\mu}_k^e &= \underline{\mu}_k^x + \Sigma_k^{xy} (\Sigma_k^{yy})^{-1} \left(\hat{\underline{y}}_k - \underline{\mu}_k^y \right) \\ \Sigma_k^e &= \Sigma_k^x - \Sigma_k^{xy} (\Sigma_k^{yy})^{-1} \Sigma_k^{yx} \ . \end{aligned}$$

Note that we implicitly assume that the state and measurement are jointly Gaussian.

B. Moments of Polynomial Transformations of Gaussian Random Variables

In this section, an efficient method for computing the exact first two moments of the polynomial transformated Gaussian random vector (5) is given. By this means, the time and measurement update step introduced in the previous section can be performed in an optimal manner.

Let $\underline{\boldsymbol{z}} \sim \mathcal{N}(\underline{z} - \underline{\mu}^z, \Sigma)$ be an n-dimensional Gaussian random vector and let

$$\underline{g}(\underline{\boldsymbol{z}}) = \begin{bmatrix} g_1(\underline{\boldsymbol{z}}) \\ \vdots \\ g_N(\underline{\boldsymbol{z}}) \end{bmatrix}$$

be an N-dimensional polynomial function. The component functions are sum of products of the form

$$g_l(\underline{\boldsymbol{z}}) = \sum_j a_{j,l} \prod_{i=1}^{n} \boldsymbol{z}_i^{s_{i,j,l}} \ ,$$

where $\boldsymbol{z}_i$ are the componponents of $\underline{\boldsymbol{z}}$, $a_{j,l}$ are coefficients, and $s_{i,j,l}$ are the exponents.

The first moment $\underline{\mu}^g$ of $g(\underline{\boldsymbol{z}})$ is given by

$$\begin{aligned} \underline{\mu}^g &= \begin{bmatrix} \underline{\mu}_1^g \\ \vdots \\ \underline{\mu}_N^g \end{bmatrix} = \mathrm{E}\big\{ g(\underline{\boldsymbol{z}}) \big\} = \begin{bmatrix} \mathrm{E}\big\{ g_1(\underline{\boldsymbol{z}}) \big\} \\ \vdots \\ \mathrm{E}\big\{ g_N(\underline{\boldsymbol{z}}) \big\} \end{bmatrix} \\ &= \begin{bmatrix} \sum_j a_{j,1} \, \mathrm{E}\big\{ \prod_{i=1}^n \boldsymbol{z}_i^{s_{i,j,1}} \big\} \\ \vdots \\ \sum_j a_{j,N} \, \mathrm{E}\big\{ \prod_{i=1}^n \boldsymbol{z}_i^{s_{i,j,N}} \big\} \end{bmatrix} \ . \end{aligned} \tag{6}$$

Analogously, the covariance matrix $\Sigma^g = (\sigma_{l,m})_{l,m=1,\dots,N}$ of $g(\underline{\boldsymbol{z}})$ turns out to be composed of

$$\sigma_{l,m} = \mathrm{E}\big\{ g_l(\underline{\boldsymbol{z}}) g_m(\underline{\boldsymbol{z}}) \big\} - \underline{\mu}_l^g \underline{\mu}_m^g \ .$$

Because the product $g_l(\underline{\boldsymbol{z}}) g_m(\underline{\boldsymbol{z}})$ is again a polynomial of the form $\sum_j a_{j,(l,m)} \prod_{i=1}^n \boldsymbol{z}_i^{s_{i,j,(l,m)}}$, we obtain

$$\sigma_{l,m} = \sum_j a_{j,(l,m)} \, \mathrm{E}\big\{ \textstyle\prod_{i=1}^n \boldsymbol{z}_i^{s_{i,j,(l,m)}} \big\} - \underline{\mu}_l^g \underline{\mu}_m^g \ . \tag{7}$$

For computing both the mean (6) and the covariance (7) of $g(\underline{\boldsymbol{z}})$, a formula for computing the expectation of products of *non-central dependent* normal random variables is required. Such a formula for central random variables is known in literature for a long time since Isserlis [27] (also known as Wick's formula in physics). However, as pointed out in [26], this formula is computationally intractable even for small exponentials, e.g., for computing $\mathrm{E}\{(\boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3\boldsymbol{z}_4\boldsymbol{z}_5)^4\}$ it would be already necessary to sum up $19!! = 654.729.075$ terms [26]. We use a computationally attractive formula recently derived in [26], where it would be only necessary to sum up 524.288 terms given in the next theorem.

Theorem II.1 *For the Gaussian distributed random vector $\underline{\boldsymbol{z}} \sim \mathcal{N}(\underline{z}-\underline{\mu}^z, \Sigma)$ and nonnegative integers s_1 to s_n, the following holds [26]:*

$$\mathrm{E}\{\textstyle\prod_{i=1}^n \boldsymbol{z}_i^{s_i}\} = \sum_{\nu_1=0}^{s_1} \cdots \sum_{\nu_n=0}^{s_n} \sum_{r=0}^{[s/2]} (-1)^{\sum_{i=1}^n \nu_i} \cdot \binom{s_1}{\nu_n} \cdots \binom{s_n}{\nu_n} \frac{\left(\frac{\underline{h}^T \Sigma \underline{h}}{2}\right)^r \left(\underline{h}^T \underline{\mu}^x\right)^{s-2r}}{r!(s-2r)!} \quad , \quad (8)$$

where $s = s_1 + \ldots + s_n$ and $\underline{h} = \left[\frac{s_1}{2} - \nu_1, \ldots, \frac{s_n}{2} - \nu_n\right]^T$.

Altogether, we have derived a fully automatic procedure for computing the first two moments of a polynomial transformation of normal random vectors that works as follows:

1) Express the first two moments of $\underline{g}(\underline{\boldsymbol{z}})$ (see (5)) as the expectation of sums of products of normal random variables (see (6) and (7)).
2) Compute the expectation of the sum of products by means of Theorem II.1 (or any other suitable formula).

Step 2 is the only computationally demanding step. However, in practical implementations simple code optimization techniques such as caching significantly improve the speed and yield an efficient procedure. Furthermore, the polynomial simplification in (7) can be performed efficiently. This simplification can also exploit knowledge about uncorrelated random variables in order to eliminate zero terms.

Note that general nonlinear systems can be handled by performing a polynomial approximation of the system functions such as a higher-order Taylor series expansion or an approximation based on Stirling's formula [14]. In case of a local polynomial approximation, it is particularly important that the higher-order moments can be calculated efficiently, as the system functions vary over time.

Besides using the above procedure for optimal online estimation, it can also be used for offline parameter tuning of other filters such as the UKF or for deriving precise Cramér-Rao lower bounds.

III. Multiple Target Tracking based on AMC

In this section, the introduced optimal Gaussian filter for polynomial systems is applied to the problem of tracking multiple targets with the Symmetric Measurement Equation (SME) filter [7], [9], [10]. The basic idea of the SME filter is to use a symmetric transformation in order construct a pseudo-measurement based on the original measurements. By this means, a new (nonlinear) measurement equation is obtained, which does not contain any data association uncertainty. If symmetric polynomials [7], [9], [10] are used for constructing the pseudo-measurements, a polynomial measurement equation is obtained, which can be handled with the method introduced in this paper.

A simple one-dimensional example illustrates the basic idea of the SME approach: Let the position of two one-dimensional targets be $\boldsymbol{x}_{k,1}$ and $\boldsymbol{x}_{k,2}$. From these targets, two position measurements $\hat{y}_{k,1}$ and $\hat{y}_{k,2}$ are received that are corrupted with additive Gaussian noise, i.e.,

$$\begin{aligned} \hat{y}_{k,\pi(1)} &= \boldsymbol{x}_{k,1} + \boldsymbol{v}_{k,1} \ , \\ \hat{y}_{k,\pi(2)} &= \boldsymbol{x}_{k,2} + \boldsymbol{v}_{k,2} \ , \end{aligned}$$

where $\pi(\cdot)$ is the unknown target-to-measurement mapping. The SME approach now removes the data association uncertainty $\pi(\cdot)$ from the measurement equation by performing a symmetric transformation such as the *Sum-Of-Powers* transformation

$$\underline{y}_k^* := T(\hat{y}_{k,\pi(1)}, \hat{y}_{k,\pi(2)}) = T(\boldsymbol{x}_{k,1} + \boldsymbol{v}_{k,1}, \boldsymbol{x}_{k,2} + \boldsymbol{v}_{k,2})$$

with

$$T(x_1, x_2) = \begin{bmatrix} x_1 + x_2 \\ x_1^2 + x_2^2 \end{bmatrix}$$

and pseudo-measurement $\underline{y}_k^*$. Due to the symmetry of $T(\cdot)$, the pseudo-measurement $\underline{y}_k^*$ can be calculated without knowing the measurement-to-target mapping $\pi(\cdot)$. Hence, the data association problem has been reformulated to a nonlinear estimation problem. Several polynomial transformations have been suggested for higher dimensions and higher target numbers [9], [10]. For instance, the *Sum-Of-Powers* transformation can be used for tracking any number of targets in two-dimensional space [9], [10]. Note that similar to the SME filter, symmetric transformations are used in [28] for multi-target tracking in a transformed space.

Because the noise in the new measurement equation is not additive, standard nonlinear estimators such as the EKF or particle filter [29] are rather difficult to apply [9], [10] to this problem. As a consequence, the UKF [9], [10] has shown to be suitable for the SME filter because it can directly be applied to this kind of measurement equation. Unfortunately the UKF turns out to have other disadvantages: First, in case of closely-spaced targets, the UKF may yield a singular covariance matrix as pointed out in [9], [10]. Furthermore, as the UKF gives poor approximations for the higher-order terms of the measurement equation, the estimation quality decreases with an increasing number of targets. The introduced method in this paper gives the optimal solution and hence, considers all higher-order moments of the measurement equation.

In the following, we consider scenarios with two and three targets performing various maneuvers. We first describe the particular settings for the simulations and then discuss the simulation results in detail.

A. System and Measurement Model of a Target

The state vector of a single target is given by $\underline{x}_{k,i} = [x_{k,i}, y_{k,i}, \dot{x}_{k,i}, \dot{y}_{k,i}]^T$, where i is the index of the associated target, consisting of the object's position $[x, y]^T$ and velocity $[\dot{x}, \dot{y}]^T$.

The SME filter estimates the full joint density of the joint target state vector

$$\underline{\boldsymbol{x}}_k = \begin{bmatrix} \underline{\boldsymbol{x}}_{k,1} \\ \vdots \\ \underline{\boldsymbol{x}}_{k,n} \end{bmatrix} \ ,$$

where n is the number of targets. In the considered scenario, each target state evolves according to a constant velocity model in two-dimensional space leading to the linear time-discrete system equation given by

$$\underline{\boldsymbol{x}}_{k+1} = \mathbf{A}_k \underline{\boldsymbol{x}}_k + \underline{\boldsymbol{w}}_k \ ,$$

where $\mathbf{A}_k = \text{diag}\{\mathbf{A}_{k,1}, \ldots, \mathbf{A}_{k,n}\}$ with

$$\mathbf{A}_{k,i} = \begin{bmatrix} 1 & 0 & \mathrm{T} & 0 \\ 0 & 1 & 0 & \mathrm{T} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ ,$$

and sampling interval T. The system noise $\underline{\boldsymbol{w}}_k$ is assumed to be zero-mean white Gaussian noise. The corresponding covariance matrix is $\mathbf{Q}_k = \mathrm{E}\{\underline{\boldsymbol{w}}_k \underline{\boldsymbol{w}}_k^{\mathrm{T}}\} = \sigma_w^2 \cdot \text{diag}\{\mathbf{Q}_{k,1}, \ldots, \mathbf{Q}_{k,n}\}$, with

$$\mathbf{Q}_{k,i} = \begin{bmatrix} \frac{1}{4}\mathrm{T}^4 & 0 & \frac{1}{2}\mathrm{T}^3 & 0 \\ 0 & \frac{1}{4}\mathrm{T}^4 & 0 & \frac{1}{2}\mathrm{T}^3 \\ \frac{1}{2}\mathrm{T}^3 & 0 & \mathrm{T}^2 & 0 \\ 0 & \frac{1}{2}\mathrm{T}^3 & 0 & \mathrm{T}^2 \end{bmatrix} \ .$$

We assume that at each time step exactly one position measurement from each single target is received. Hence, the measurement equation is given by

$$\underline{\boldsymbol{y}}_k = \mathbf{H}_k \cdot \underline{\boldsymbol{x}}_k + \underline{\boldsymbol{v}}_k \ ,$$

where $\mathbf{H}_k = \text{diag}\{\mathbf{H}_{k,1}, \ldots, \mathbf{H}_{k,n}\}$ and

$$\mathbf{H}_{k,i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \ .$$

Assuming white Gaussian measurement noise $\underline{\boldsymbol{v}}_k$, the covariance matrix of $\underline{\boldsymbol{v}}_k$ is $\mathbf{R}_k = \text{diag}\{\mathbf{r}_{k,1}, \ldots, \mathbf{r}_{k,n}\}$ with $\mathbf{r}_{k,i} = \text{diag}\{\sigma_v^2, \sigma_v^2\}$.

B. Pseudo-Measurement Function

In the following, we describe the symmetric polynomials used for obtaining the pseudo-measurement function. For the sake of simplicity, we omit the time index k in the following. For two targets with joint measurement vector $\underline{y} = [x_1, x_2, y_1, y_2]^{\mathrm{T}}$, the pseudo-measurement $\underline{y}^*$ is obtained with the so-called *Sum-of-Powers* pseudo-measurement function $T(\underline{y})$ given by [9], [10]

$$T(\underline{y}) = \begin{bmatrix} \Re(p_1 + p_2) \\ \Im(p_1 + p_2) \\ \Re(p_1^2 + p_2^2) \\ \Im(p_1^2 + p_2^2) \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ x_1^2 + x_2^2 - y_1^2 - y_2^2 \\ 2 \cdot x_1 y_1 + 2 \cdot x_2 y_2 \end{bmatrix} \ ,$$

where $p_i = x_i + j \cdot y_i$ and j is the imaginary unit. The symbols $\Re$ and $\Im$ denote the real and imaginary part of a number. For the three target case with joint state vector $\underline{y} = [x_1, x_2, x_3, y_1, y_2, y_3]^{\mathrm{T}}$, the pseudo-measurement function $T(\underline{y})$ is given by

$$T(\underline{y}) = \begin{bmatrix} \Re(p_1 + p_2 + p_3) \\ \Im(p_1 + p_2 + p_3) \\ \Re(p_1^2 + p_2^2 + p_3^2) \\ \Im(p_1^2 + p_2^2 + p_3^2) \\ \Re(p_1^3 + p_2^3 + p_3^3) \\ \Im(p_1^3 + p_2^3 + p_3^3) \end{bmatrix} \ .$$

C. Using Optimal Gaussian Filtering

Using the optimal Gaussian filter based on analytic moment calculation (AMC) introduced in this work is straightforward for the SME filter as the final measurement equation

$$\underline{y}_k^* = T(\underline{y}_k) = T(\mathbf{H}_k \cdot \underline{\boldsymbol{x}}_k + \underline{\boldsymbol{v}}_k)$$

is already in polynomial form.

According to Section II-B, the moments required for (6) can be calculated in closed form for the measurement update. Furthermore, as described in [17], the complexity for calculating Σ_k^{xy} can be further reduced as the velocity and measurement noise are independent.

D. Parameter Setup

Simulations consist of three different scenarios. For each scenario, 50 Monte-Carlo trails were performed. *Scenario 1* starts with two targets and medium measurement noise level. *Scenarios 2* and *3* have been performed with medium and high measurement noise level. Both scenarios were developed to stress the filter. The associated parameter values for medium and high noise levels are listed in Table I.

Noise	Type σ_w	Type σ_v	Scenario 1	Scenario 2	Scenario 3
Medium	0.76	0.5	x	x	x
High	0.76	0.7		x	x

Table I: Noise Parameters.

E. Two Targets

The first scenario starts with two targets that move parallel for a certain time, cross each other, and then diverge. Relative to the measurement noise, the target distance is high enough from each other, so that no track switching occurs and both filters perform well. The detailed results are shown in Fig. 1. Fig. 1a illustrates the RMSE for every filter step comparing the UKF with the AMC method. Example runs for UKF and AMC are plotted in Fig. 1c and Fig. 1d.

The second scenario is identical to the first scenario except for the distance between both targets, which is reduced to stress the filters. The results depicted in Fig. 2 and Fig. 3 show that the optimal Gaussian filter outperforms the UKF. Example runs concerning each filter can be seen in Fig. 2

and Fig. 3. The main reason for the better performance of the optimal Gaussian filter is that the covariance matrix estimated by the UKF is not positive definite anymore when the targets are crossing. This phenomenon is also discussed in [9], [10].

F. Three Targets

The third scenario consists of three targets moving close to each other. Over the course, the targets come closer, cross, and then diverge. As in the second scenario, the AMC outperforms the UKF which is illustrated in Fig. 4 and Fig. 5. Track switching occurs for the UKF and cannot be observed for the AMC method.

IV. CONCLUSIONS

In this work, an efficient method for performing optimal Gaussian filtering for polynomial systems has been proposed. For this purpose, the method exploits the efficient calculation of the expectation of products of Gaussian random variables [26]. In doing so, the obtained filter is tractable even for high dimensions and higher-order polynomial system functions. The filter was applied to a multi-target tracking scenario, where the origin of position measurements is uncertain. In order to remove this data association uncertainty, the measurement equation is transformed with the help of a symmetric polynomial function [9], [10]. A comparison with the UKF shows the benefit of the optimal filter in the tracking scenario.

Besides of online application of the introduced filter, a further application would be to calculate precise Cramér-Rao lower bounds with the help of analytic moment calculation for multi-target tracking (see [9], [10]).

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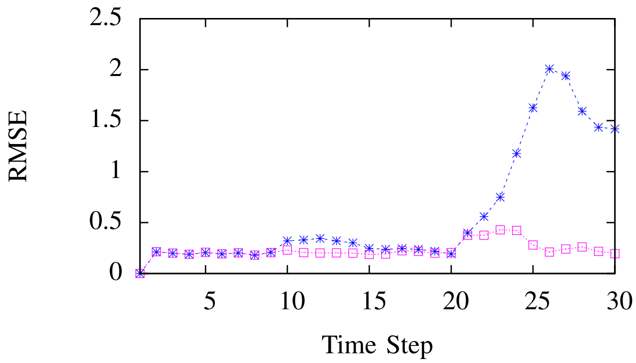
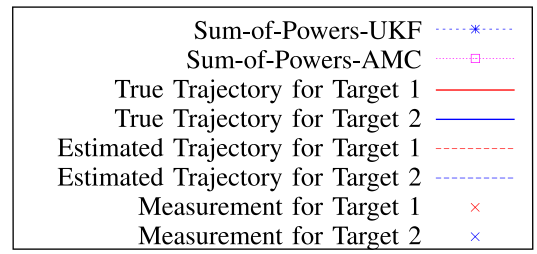
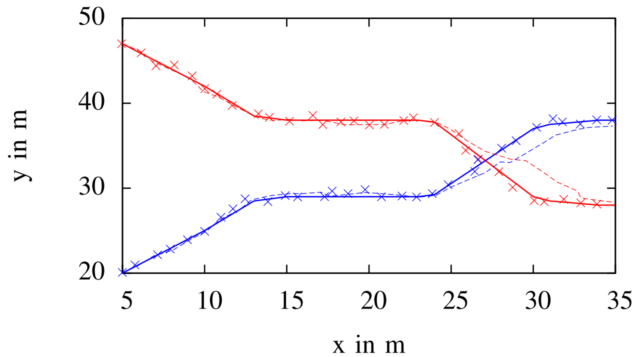


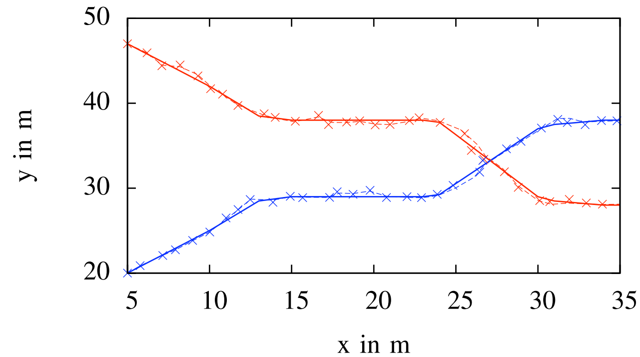
(a) RMSE for medium noise.



(b) Legend.

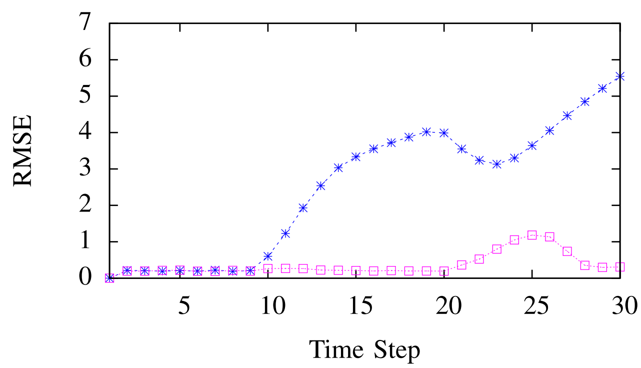


(c) Example run (noise level: medium), UKF.

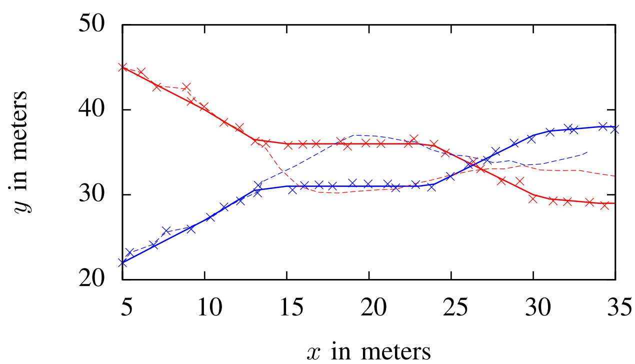


(d) Example run (noise level: medium), AMC.

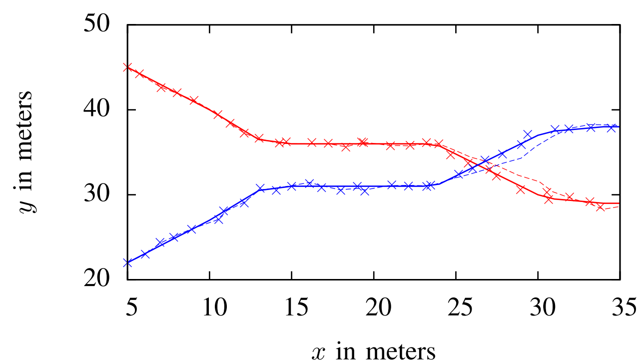
Figure 1: Two well separated targets (noise level: medium).



(a) RMSE for medium noise.

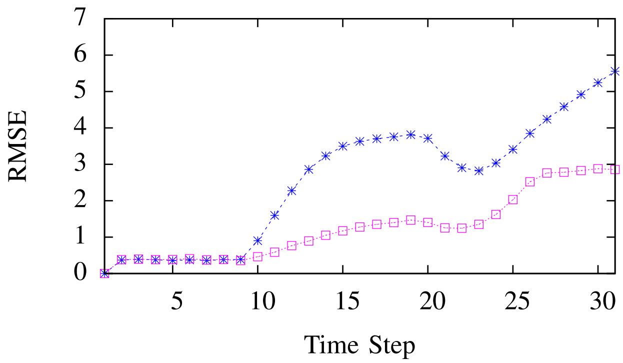


(b) Example run (noise level: medium), UKF.

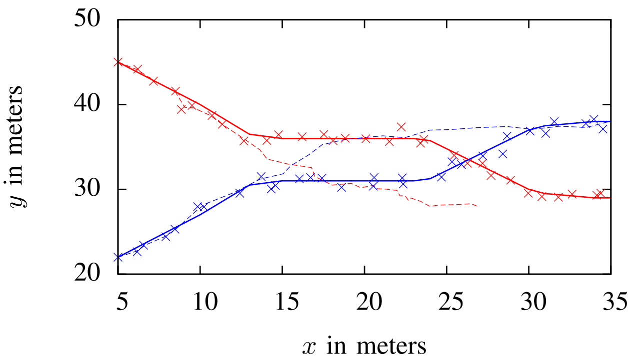


(c) Example run (noise level: medium), AMC.

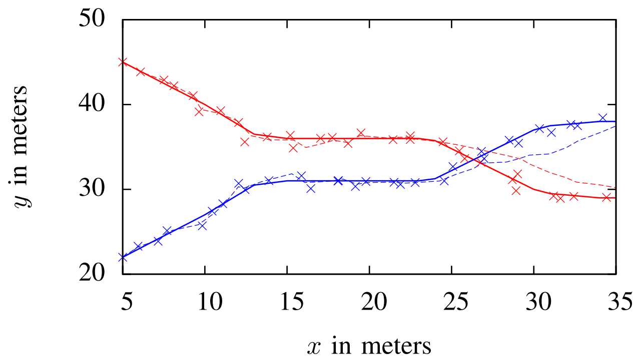
Figure 2: Two closely-spaced targets (noise level: medium).



(a) RMSE for high noise.

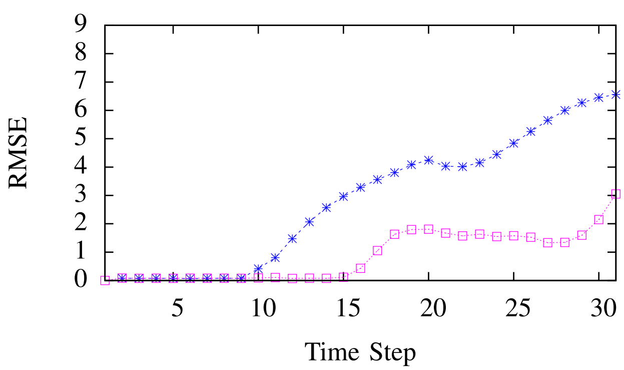


(b) Example run (noise level: high), UKF.

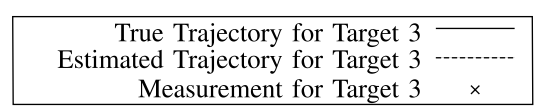


(c) Example run (noise level: high), AMC.

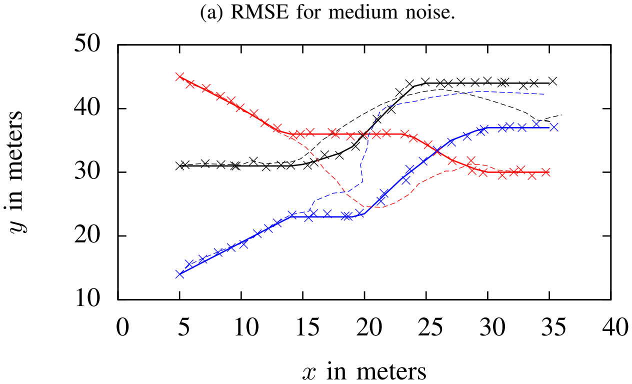
Figure 3: Two closely-spaced targets (noise level: high).



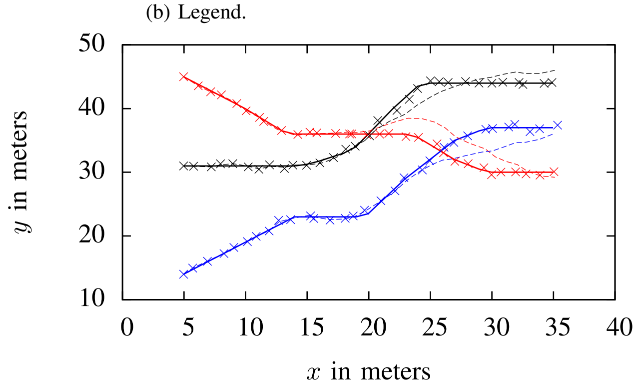
(a) RMSE for medium noise.



(b) Legend.

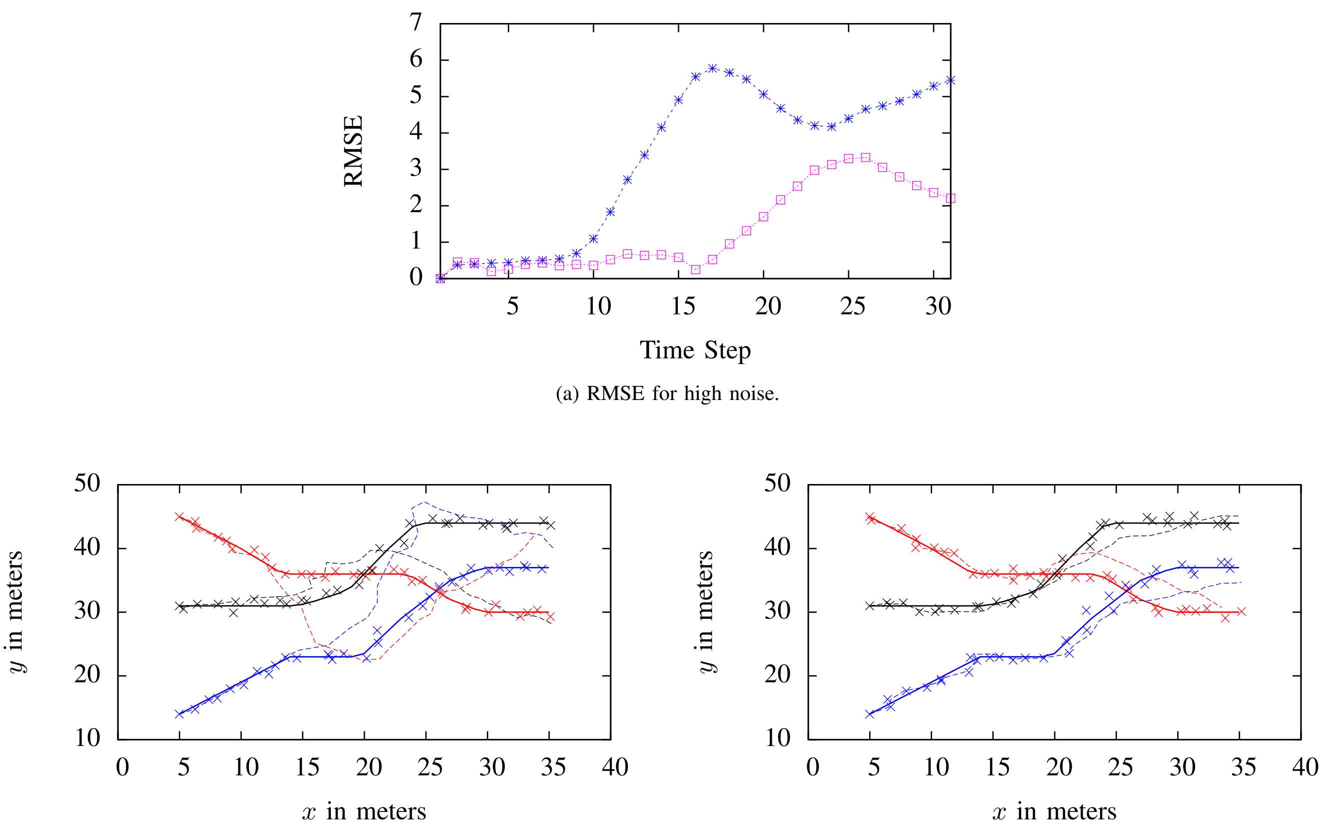


(c) Example run (noise level: medium), UKF.



(d) Example run (noise level: medium), AMC.

Figure 4: Three closely-spaced targets (noise level: medium).



(a) RMSE for high noise.

(b) Example run (noise level: high), UKF.

(c) Example run (noise level: high), AMC.

Figure 5: Three closely-spaced targets (noise level: high).