Modeling the Target Extent with Multiplicative Noise

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***Abstract*—Extended target tracking deals with simultaneously tracking the shape and the kinematic parameters of a target. In this work, we formulate the extended target tracking problem as a state estimation problem with both multiplicative and additive measurement noise. In case of extended targets with known orientation, we show that the best linear estimator is not consistent and, hence, is unsuitable for this problem. In order to overcome this issue, we propose a quadratic estimator for a recursive closed-form measurement update. Simulations demonstrate the performance of the estimator.**

I. Introduction

Target tracking treats the problem of recursively estimating the kinematic state, e.g., position and velocity, of a target object based on noisy measurements [1]. Usually, the target is modeled as a mathematical point without any extent. However, in many applications, this assumption is not justified and a target may cause a varying number of measurements from different spatially distributed measurement sources. Example scenarios can be found in surveillance and robotics when the resolution capability of the sensor is higher than the spatial extent of the target.

In this paper, the basic idea is to estimate a shape approximation of the extended target in addition to its kinematic state [2]–[4]. In this manner, it is not necessary to explicitly estimate the locations of measurement sources.

A main challenge is that the measurements are corrupted by two different sources of uncertainty, i.e., the uncertainty about the measurement source and the measurement noise. Additionally, the shape parameters of the target are unknown and part of the state to be estimated. As a consequence, a hierarchical model is obtained, for which Bayesian inference is generally very tedious.

A. Contributions

The main contribution of this work is the formulation of a subclass of spatial distribution models [4], [5] as an explicit measurement equation corrupted with multiplicative noise. By this means, standard nonlinear filtering techniques can be applied for tracking the extent of a target. In particular, we derive a quadratic estimator, which allows for a closed-form recursive measurement update. By this means, no significant approximations have to be performed and the use of particle filtering techniques is avoided at all. The approach can be used for tracking circular shapes, elliptic shapes, and stick targets.

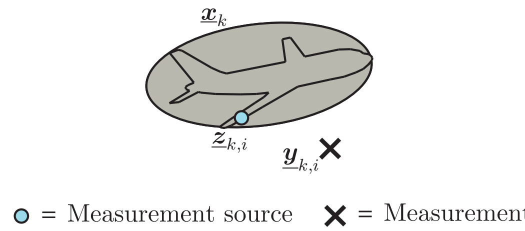


Fig. 1: Extended target and its approximation with an ellipse. The target state including shape parameters is given by $\underline{\boldsymbol{x}}_k$. The i-th measurement source at time step k is $\underline{\boldsymbol{z}}_{k,i}$ and the corresponding measurement is denoted with $\underline{\boldsymbol{y}}_{k,i}$.

B. Related Work

Spatial distributions [4], [5] assume that each measurement source is an independent random draw from an object-dependent probability distribution. Spatial distributions are a very general concept, which has been used, e.g., for tracking stick targets, Gaussian mixtures, and circles [4]–[7]. In [8], [9], spatial distributions have been embedded into *Probability Hypothesis Density (PHD)* filters for tracking multiple extended objects. As spatial distribution yield a high-dimensional nonlinear hierarchical estimation problem, particle filter methods [10] are mainly used for an approximate Bayesian measurement update.

In [3], [11], an elliptic target extent is modeled with Gaussian spatial distribution and the uncertainty about the elliptic extent is expressed by means of a random symmetric positive definite matrix. Random matrices have been used for tracking multiple extended objects within the *Probabilistic Multiple-Hypothesis Tracker (PMHT)* framework [12] in [13] and a hybridization solution was proposed in [14]. In [15], the random matrix approach has been adopted for direct measurements of the principal components.

An alternative approach called *Random Hypersurface Model* (*RHM*) has been introduced in [16]. An *RHM* assumes that each measurement source lies on a scaled version of the shape boundary. *RHMs* provide a systematic way to model different target shapes from ellipses [17] to arbitrary star-convex shapes [18].

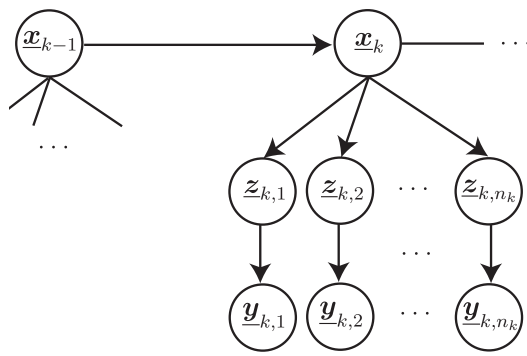


Fig. 2: Graphical model for spatial distributions.

C. Overview

The remainder of this paper is structured as follows. In Section II, a detailed problem formulation and description of spatial distribution models is given. Then, we show how particular spatial distribution models can be formulated as a measurement equation with multiplicative measurement noise (Section III). Based on this measurement equation, we introduce a quadratic estimator in Section IV for simultaneously tracking and estimating the shape of the target. This estimator is evaluated in Section V. The conclusions are given in Section VI.

II. Problem Formulation

We consider the tracking of a single extended object. The state parameters of the extended object are modeled as a random vector $\underline{\boldsymbol{x}}_k$ with time index k. Note that the state vector $\underline{\boldsymbol{x}}_k$ consists of variables for the position, velocity, or acceleration and also of parameters for the target shape.

An illustration of the involved random variables is given in Figure 1 and a graphical model is depicted in Figure 2.

Remark 1. In this work, we focus on estimating the parameters of a single extended target. Several extensions of spatial distribution models to clutter and multiple targets have been proposed in literature [4], [5], [8], [9]. Of course, the estimator for a single extended target, which is proposed in this work, can be embedded into these approaches.

A. Measurement Model

At each time step k, a set of n_k measurements

$$\mathcal{Y}_k = \{\underline{\boldsymbol{y}}_{k,1}, \ldots, \underline{\boldsymbol{y}}_{k,n_k}\}$$

from the extended target becomes a available. The measurement generation process for a measurement $\underline{\boldsymbol{y}}_{k,i}$ consists of two parts: The target extent model describes where a measurement sources $\underline{\boldsymbol{z}}_{k,i}$ lies on the target. It is specified by a spatial distribution [4], [5] $p(\underline{z}_{k,i}|\underline{x}_k)$, which depends on the target parameters, e.g., position, length, and orientation of the target. The sensor model specifies the measurement $\underline{\boldsymbol{y}}_{k,i}$ arising from the measurement source and it is characterized by the conditional density $p(\underline{y}_{k,i}|\underline{x}_k)$.

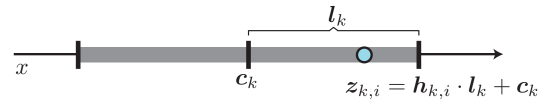


Fig. 3: Modeling the target extent with multiplicative noise: Stick target with center $\boldsymbol{c}_k$ and length $2\boldsymbol{l}_k$ in one-dimensional space. A measurement source is given by $\boldsymbol{z}_{k,i} = \boldsymbol{h}_{k,i} \cdot \boldsymbol{l}_k + \boldsymbol{c}_k$, where $\boldsymbol{h}_{k,i}$ is a multiplicative noise term that scales the length $\boldsymbol{l}_k$ in order reach each point on the stick.

In this work, we focus on Cartesian position measurements of the measurement source, i.e., $p(\underline{y}_{k,i}|\underline{x}_k)$ results from the following measurement equation

$$\underline{\boldsymbol{y}}_{k,i} = \underline{\boldsymbol{z}}_{k,i} + \underline{\boldsymbol{v}}_{k,i} \ , \tag{1}$$

which maps the measurement source $\underline{\boldsymbol{z}}_{k,i}$ and an additive noise term $\underline{\boldsymbol{v}}_{k,i}$ to the measurement $\underline{\boldsymbol{y}}_{k,i}$. Note that almost all relevant sensors can be formalized in this manner, e.g., angle-distance measurements can be transformed to Cartesian measurements.

B. Dynamic Model

The temporal evolution of the state is modeled as a Markov model characterized by a conditional density function $p(\underline{x}_{k+1}|\underline{x}_k)$. For example, $p(\underline{x}_{k+1}|\underline{x}_k)$ may be specified by a linear system equation

$$\underline{\boldsymbol{x}}_{k+1} = \mathbf{A}_k \underline{\boldsymbol{x}}_k + \underline{\boldsymbol{w}}_k \ , \tag{2}$$

where $\mathbf{A}_k$ is the system matrix and $\underline{\boldsymbol{w}}_k$ is an additive noise term.

III. Modeling Extended Targets As Multiplicative Measurement Noise

In this section, we show that the target extent model, i.e., a spatial distribution, can be represented as a measurement equation corrupted by multiplicative noise.

A. Basic Idea

In order to illustrate the basic idea, we restrict ourselves to a one-dimensional extended target, whose length $\boldsymbol{l}_k$ and center $\boldsymbol{c}_k$ are to be estimated, i.e., the state is given by $\underline{\boldsymbol{x}}_k = \left[\boldsymbol{c}_k, \boldsymbol{l}_k\right]^T$. We aim at a measurement source equation, which maps the state vector $\underline{\boldsymbol{x}}_k$ to a measurement source $\boldsymbol{z}_{k,i}$ on the target.

The key observation (see Figure 3) is that each point on the target can be reached by scaling the length $\boldsymbol{l}_k$ and adding the center $\boldsymbol{c}_k$, i.e.,

$$\boldsymbol{z}_{k,i} = \boldsymbol{h}_{k,i} \cdot \boldsymbol{l}_k + \boldsymbol{c}_k \ , \tag{3}$$

where $\boldsymbol{h}_{k,i}$ is a multiplicative noise term. The noise $\boldsymbol{h}_{k,i}$ specifies the measurement source on the target. It should be an element of the interval $[-1, 1]$ and its probability distribution can be independent of the state.

The probability density of $\boldsymbol{h}_{k,i}$ can be interpreted as a reference spatial distribution

$$p(h_{k,i}) := p(z_{k,i}|l_k = 1, c_k = 0) \ ,$$

which specifies the spatial distribution of a measurement source $z_{k,i}$ for length $l_k = 1$ and center $c_k = 0$. For given l_k and c_k, the measurement source equation (3) specifies the spatial distribution

$$p(z_{k,i}|l_k, c_k) = \frac{1}{l_k} \cdot p_h\left(\frac{z_{k,i} - c_k}{l_k}\right) \ ,$$

where $p_h(\cdot)$ is a shorthand for $p(h_{k,i})$. This is a scaled and translated version of the reference spatial distribution $p(h_{k,i})$.

With the help of the sensor model (1), we obtain the final measurement equation

$$\boldsymbol{y}_{k,i} = \boldsymbol{h}_{k,i} \cdot \boldsymbol{l}_k + \boldsymbol{c}_k + \boldsymbol{v}_{k,i} \quad , \tag{4}$$

which relates the state $\underline{\boldsymbol{x}}_k = \left[\boldsymbol{c}_k, \boldsymbol{l}_k\right]^T$ to the observed measurement $\boldsymbol{y}_{k,i}$.

Example 1. A simple example for a reference distribution is the standard normal distribution

$$p(h_{k,i}) = N(h_{k,i} - 0, 1) \ .$$

In this case, the spatial distribution is a normal distribution with mean c_k and standard deviation l_k, i.e.,

$$p(z_{k,i}|l_k, c_k) = N(z_{k,i} - c_k, l_k^2) \ .$$

Example 2. When the reference distribution is a uniform distribution on the interval $[-1, 1]$ given by

$$p(h_{k,i}) = U(h_{k,i} - 0, 1) \ ,$$

the spatial distribution is a uniform distribution with center a c_k and length $2\ l_k$

$$p(z_{k,i}|l_k, c_k) = U(z_{k,i} - c_k, l_k) \ .$$

B. Axis-Aligned Extended Targets

A multivariate extension of the multiplicative measurement equation (4) can be obtained for axis-aligned extended objects by using (4) for each dimension. In this case, the state for an n-dimensional extended target consists of the center $\underline{\boldsymbol{c}}_k \in \mathbb{R}^N$ and a vector $\underline{\boldsymbol{l}}_k \in \mathbb{R}^N$ for the length in each dimension

$$\underline{\boldsymbol{x}} = \left[\underline{\boldsymbol{c}}_k^T, \underline{\boldsymbol{l}}_k^T\right]^T \in \mathbb{R}^{2N} \ .$$

The multivariate final measurement equation then turns out to be

$$\underline{\boldsymbol{y}}_{k,i} = \mathbf{H}_{k,i} \cdot \underline{\boldsymbol{l}}_k + \underline{\boldsymbol{c}}_k + \underline{\boldsymbol{v}}_{k,i} \ , \tag{5}$$

where $\mathbf{H}_{k,i} = \mathrm{diag}(\boldsymbol{h}_{k,i}^{(1)}, \ldots, \boldsymbol{h}_{k,i}^{(N)}) \in \mathbb{R}^{N \times N}$ is a multiplicative noise matrix consisting of the random variables $\boldsymbol{h}_{k,i}^{(j)}$. Note that $\boldsymbol{h}_{k,i}^{(j)}$ do not have to be independent.

Example 3. If the components of $\mathbf{H}_{k,i}$ are standard normal distributed, i.e., $p(h_{k,i}^{(j)}) = N(h_{k,i}^{(j)} - 0, 1)$, then $p(\underline{h}_{k,i}) = N(\underline{h}_{k,i} - 0, \mathbf{I}_n)$ is the multivariate standard normal distribution, where $\mathbf{I}_n$ is the n-dimensional identity matrix. Hence, $p(\underline{z}_{k,i}|\underline{l}_k, \underline{c}_k)$ is a multivariate axis-aligned normal distribution with covariance matrix $\mathrm{diag}(\underline{l}_k)^2$ and center $\underline{c}_k$. This can also be interpreted as an elliptic target shape, whose principal components are given by $\underline{l}_k$. This is essentially the same spatial distribution as in [3], however, the parameterization is different.

Example 4. When $p(h_{k,i}^{(j)}) = U(h_{k,i}^{(j)} - 0, 1)$, the vector $\underline{l}_k$ specifies an axis-aligned rectangular uniform distribution.

C. Incorporating the Orientation

Usually, the orientation of the target is unknown and shall be estimated, too. In this work, we assume that the target is aligned along its velocity vector, which is a realistic assumption in many applications. For this purpose, the state vector also consists of the velocity vector $\underline{\boldsymbol{c}}_k^v$, i.e., $\underline{\boldsymbol{x}}_k = \left[\boldsymbol{c}_k^T, \boldsymbol{l}_k^T, (\underline{\boldsymbol{c}}_k^v)^T\right]^T$. In this case, we obtain the following measurement equation

$$\underline{\boldsymbol{y}}_{k,i} = \mathbf{R}(\underline{\boldsymbol{c}}_k^v) \cdot \mathbf{H}_{k,i} \cdot \underline{\boldsymbol{l}}_k + \underline{\boldsymbol{c}}_k + \underline{\boldsymbol{v}}_{k,i} \ , \tag{6}$$

where $\mathbf{R}(\underline{\boldsymbol{c}}_k^v)$ denotes the rotation matrix corresponding to the velocity vector $\underline{\boldsymbol{c}}_k^v$.

Example 5. In two-dimensional space

$$\mathbf{R}(\underline{\boldsymbol{c}}_k^v) = \frac{1}{||\underline{\boldsymbol{c}}_k^v||} \begin{bmatrix} \boldsymbol{c}_k^{v,1} & -\boldsymbol{c}_k^{v,2} \\ \boldsymbol{c}_k^{v,1} & \boldsymbol{c}_k^{v,2} \end{bmatrix} \ ,$$

where $\underline{\boldsymbol{c}}_k^v = \left[\boldsymbol{c}_k^{v,1}, \boldsymbol{c}_k^{v,2}\right]^T$.

The term $\mathbf{R}(\underline{\boldsymbol{c}}_k^v) \cdot \mathbf{H}_{k,i}$ can be interpreted as state-dependent multiplicative noise. Because $\mathbf{R}(\underline{\boldsymbol{c}}_k^v)$ is a nonlinear term, the following approximation may be suitable.

Approximation 1. $\mathbf{R}(\underline{\boldsymbol{c}}_k^v)$ is substituted with $\mathbf{R}(\hat{\underline{c}}_k^v)$, where $\hat{\underline{c}}_k^v$ is the latest estimate.

Due to Approximation (1), $\mathbf{R}(\hat{\underline{c}}_k^v)\mathbf{H}_{k,i}$ is again state independent multiplicative noise.

Remark 2. Note that it would be possible to estimate an orientation that is not aligned with the velocity direction.

IV. Recursive Tracking Algorithm

A recursive tracking algorithm consists of a *measurement update* and a *time update* step. In order to perform the measurement update, we first note that both measurement equations (5) and (6) can be written in the following form

$$\underline{\boldsymbol{y}}_{k,i} = \mathbf{F}_{k,i} \cdot \underline{\boldsymbol{x}}_k + \underline{\boldsymbol{v}}_{k,i} \ , \tag{7}$$

where $\mathbf{F}_{k,i}$ is a multiplicative noise term, i.e., a random matrix, and $\underline{\boldsymbol{v}}_{k,i}$ Gaussian additive noise.

For example, for axis-aligned extended targets (5), $\mathbf{F}_{k,i} := \begin{bmatrix}\mathbf{H}_{k,i} & \mathbf{I}_n\end{bmatrix}$, where $\mathbf{I}_n$ is the identity matrix with dimension n.

The current state estimate given the measurements $\mathcal{Y}_1, \ldots, \mathcal{Y}_{k-1}$ and $\underline{\boldsymbol{y}}_{k,1}, \ldots, \underline{\boldsymbol{y}}_{k,i-1}$ is denoted with $\hat{\underline{x}}_{k,i-1}$ and the corresponding covariance matrix is $\mathbf{C}_{k,i-1}$. The measurement update step takes the measurement $\underline{\boldsymbol{y}}_{k,i}$ and determines the updated estimate

$$\hat{\underline{x}}_{k,i} \quad , \text{ and } \quad \mathbf{C}_{k,i} \ .$$

The time update predicts the last estimate of time step k, i.e., $\hat{\underline{x}}_{k,n_k}$ and $\mathbf{C}_{k,n_k}$ to the next time step with the help of the dynamic model. The predicted estimate is denoted with $\hat{\underline{x}}_{k+1,0}$ and $\mathbf{C}_{k+1,0}$. In case of a linear model such as (2), the time update can be performed with the standard Kalman filtering formulas [19]. In the following, we focus on the measurement update.

A. Linear Minimum Mean Squared (LMMSE) Estimator

The Linear Minimum Mean Squared Error (LMMSE) estimator for systems with (state-independent) multiplicative measurement noise such as (7) is known for a long time [20], [21]. However, the LMMSE estimator for estimating the target extent is not consistent, i.e., for static systems, the estimate does not converge to the true value with an increasing number of measurements. In fact, this becomes already apparent for the one-dimensional case, when looking at equation (3) and noticing that $\boldsymbol{y}_{k,i}$ is uncorrelated with $\boldsymbol{l}_k$ if the mean of the multiplicative noise $\boldsymbol{h}_{k,i}$ is zero.

In order to show the inconsistency of the LMMSE estimator in general, we rewrite (7) to

$$\underline{\boldsymbol{y}}_{k,i} = \bar{\mathbf{F}}_{k,i} \cdot \underline{\boldsymbol{x}}_k + \underbrace{\tilde{\mathbf{F}}_{k,i} \cdot \underline{\boldsymbol{x}}_k + \underline{\boldsymbol{v}}_{k,i}}_{=:\underline{\boldsymbol{v}}^*_{k,i}} \ , \tag{8}$$

where $\bar{\mathbf{F}}_{k,i} := \mathrm{E}\{\mathbf{F}_{k,i}\}$ is defined as the mean matrix of $\mathbf{F}_{k,i}$ and $\tilde{\mathbf{F}}_{k,i} := \mathbf{F}_{k,i} - \bar{\mathbf{F}}_{k,i}$.

As proven in [20], the term $\underline{\boldsymbol{v}}^*_{k,i}$ is uncorrelated to $\underline{\boldsymbol{x}}_k$ in case of state-independent $\tilde{\mathbf{F}}_{k,i}$. For this reason, the Kalman filter equations yield the LMMSE estimator for the system (8).

In order to show that the LMMSE estimator is inconsistent, we consider an axis-aligned extended target according to measurement equation (5), i.e., $\bar{\mathbf{F}}_{k,i} = \begin{bmatrix}\bar{\mathbf{H}} & \mathbf{I}_n\end{bmatrix}$, where $\bar{\mathbf{H}}$ is a diagonal matrix.

Furthermore, we restrict ourselves to a static extended object, i.e., for system matrix $\mathbf{A}_k = \mathbf{I}_k$ and the state to be estimated consists of the center and length, i.e., $\underline{\boldsymbol{x}}_k = [\underline{\boldsymbol{c}}_k^T, \underline{\boldsymbol{l}}_k^T]^T$. In this case, the rank of the observability matrix for linear systems [19] is

$$\text{rank}\left(\begin{bmatrix}\bar{\mathbf{F}}_{k,1} \\ \vdots \\ \bar{\mathbf{F}}_{k,n_k}\end{bmatrix}\right) = N < dim(\underline{x}_k) = 2\,N \ .$$

Hence, it is not possible to estimate *both* the center and length of an extended object with a linear estimator. Nevertheless, if the length is known, the linear estimator is feasible.

Note that this is an important insight as it contains many relevant special cases. For example, it says that it is not possible to estimate both the mean and standard deviation of a normal distribution with an LMMSE estimator (see also Example 1).

B. Quadratic Estimator

Because linear estimators are unsuitable for extended targets modeled as multiplicative noise, we suggest to use a quadratic estimator as described in [22], [23]. According to [22], [23] the best quadratic estimator can be obtained by considering the extended system

$$\underline{\boldsymbol{Y}}_{k,i} := \begin{bmatrix}\underline{\boldsymbol{y}}_{k,i} \\ \underline{\boldsymbol{y}}_{k,i}^{[2]}\end{bmatrix} = \begin{bmatrix}\mathbf{F}_{k,i} \cdot \underline{\boldsymbol{x}}_k + \underline{\boldsymbol{v}}_{k,i} \\ (\mathbf{F}_{k,i} \cdot \underline{\boldsymbol{x}}_k + \underline{\boldsymbol{v}}_{k,i})^{[2]}\end{bmatrix} \ , \tag{9}$$

where $\mathbf{F}_{k,i}$ is multiplicative noise, i.e., a random matrix and $\underline{\boldsymbol{v}}_{k,i}$ is additive Gaussian noise. The operator $(\cdot)^{[r]}$ in (9) is defined recursively for all vectors $\underline{a}$ with $\underline{a}^{[0]} = \underline{a}$ and $\underline{a}^{[r]} = \underline{a}^{[r-1]} \otimes \underline{a}$, where $\otimes$ denotes the Kronecker product.

Definition 1 (Kronecker Product)**.** The Kronecker product of two matrices $\mathbf{M} \in \mathbb{R}^{p\times q}$ and $\mathbf{N} \in \mathbb{R}^{s\times t}$ is defined as

$$\mathbf{M} \otimes \mathbf{N} := \begin{bmatrix} m_{1,1}\mathbf{N} & \dots & m_{1,s}\mathbf{N} \\ \vdots & \ddots & \vdots \\ m_{p,1}\mathbf{N} & \dots & m_{p,q}\mathbf{N}\end{bmatrix} \in \mathbb{R}^{(p\cdot s)\times(q\cdot t)} \ .$$

The best quadratic estimator is obtained by a linear projection [22], [23] of $\underline{\boldsymbol{x}}_k$ onto the extended measurements $\underline{\boldsymbol{Y}}_k$, which is given by the Kalman filter formulas

$$\begin{aligned}
\hat{\underline{x}}_{k,i} &= \hat{\underline{x}}_{k,i-1} + \mathbf{C}^{Y,x}_{k,i}\left(\mathbf{C}^{Y,Y}_{k,i}\right)^{-1}\left(\hat{\underline{Y}}_{k,i} - \mathrm{E}\{\underline{\boldsymbol{Y}}_{k,i}\}\right) \\
\mathbf{C}_{k,i} &= \mathbf{C}_{k,i-1} - \mathbf{C}^{Y,x}_{k,i}\left(\mathbf{C}^{Y,Y}_{k,i}\right)^{-1}\mathbf{C}^{Y,x}_{k,i} \ ,
\end{aligned}$$

where $\mathbf{C}^{Y,x}_{k,i}$ denotes the cross-covariance matrix of the state $\underline{\boldsymbol{x}}_{k,i-1}$ and the extended measurement vector $\underline{\boldsymbol{Y}}_{k,i}$. The term $\mathbf{C}^{Y}_{k,i}$ is the covariance of the extended measurement vector and $\mathrm{E}\{\underline{\boldsymbol{Y}}_{k,i}\}$ is the predicted extended measurement.

Because (9) is a quadratic equation, the moments $\mathrm{E}\{\underline{\boldsymbol{Y}}_{k,i}\}$ $\mathbf{C}^{Y,x}_{k,i}$ and $\mathbf{C}^{Y}_{k,i}$ in (10) can be calculated efficiently in closed form according to the procedure described in [2]. In order to calculate the moments, Approximation 1 has to be used if the orientation should be incorporated (see Section III-C). Furthermore, an approximate update based on (9) can be performed with any Gaussian state estimator such as the Unscented Kalman Filter (UKF) [24]. In this case, Approximation 1 is not required.

C. Comparison

In the following, we briefly discuss the differences of the above estimator to related approaches.

Random Matrix Approach: The random matrix approach [3] is based on a neat representation of an uncertain ellipse with an (Inverse-) Wishart density for symmetric semi-positive definite (SPD) matrices. An Inverse-Wishart density can be characterized by means of SPD Matrix plus a one-dimensional parameter α specifying its uncertainty. The approach presented in this work can also be used for ellipses as described in Example 3. In this sense, the mean and covariance of the principal components represent the uncertainty of the ellipse. Hence, the uncertainty of an n-dimensional ellipse is specified with the mean and covariance matrix of an $n+1$ dimensional random vector.

Random Hypersurface Model: A Random Hypersurface Model (RHM) is based on the assumption that a measurement source lies on a scaled version of the shape contour. The approach presented here uses a scaling factor for each dimension in order to get a spatial distribution.

V. Evaluation

In this section, we present an evaluation of the proposed quadratic estimator for extended targets modeled as multiplicative noise.

A. Stationary Target in 1D

First, we consider the estimation of the center and length of a one-dimensional stationary target, i.e., the target does not move. The target extent is modeled with a normal spatial distribution as in Example 3.

Remark 3. The normal distribution can be used as an approximation of a uniform distribution by means of moment matching. Hence, this problem can be interpreted as the estimation of the endpoints of a uniform distribution, while the measurements are noise corrupted. In case of no measurement noise, this is a well-known problem in the statistics literature as it is a counterexample to maximum likelihood estimation [25]. Besides, this problem also has a variety of applications, e.g., in computer vision.

The target is located at $c = 1$ and its length is $l = 3$.[1] Hence, the spatial distribution is a normal distribution with mean 1 and standard deviation 3. The additive measurement noise (see Equation (1)) is zero-mean Gaussian with variance 3. The goal is to estimate the parameters $\underline{x} := \left[c, l\right]^T$ based on sequentially arriving measurements $\underline{y}_k$, i.e., in this case $n_k = 1$.

In the first scenario, an a priori probability density for $\underline{x}$ is given by a Gaussian distribution with covariance matrix $\mathrm{diag}([1.8, 0.6])$, i.e., the length and center are *uncorrelated*.

In the second scenario, the length and center are a priori *correlated*, i.e., $\underline{x}$ is given by a Gaussian distribution with covariance matrix $\begin{pmatrix} 0.9 & 0.6 \\ 0.6 & 0.5 \end{pmatrix}$.

For both scenarios, we compare the following four estimators:

- A *grid filter*, i.e., the probability density for $\underline{x}_k$ is discretized. By this means, it is possible to approximate the exact Bayes filter arbitrarily well. Hence, the resulting estimation error can be seen as a lower bound.
- The *LMMSE estimator* as described in [20] and discussed in Section IV-A.
- The *random matrix approach* for one-dimensional states as described in [3]. In this case, the inverse-Wishart density becomes an inverse Gamma distribution. Moment matching is used to calculate the parameters of the prior inverse-Gamma density based on the Gaussian prior.
- The *quadratic estimator* introduced in Section IV.

Figure 4 shows the root mean squared error (RMSE) of the position and length for scenario 1. It can be seen that the random matrix approach and the quadratic estimator yield comparable results, which are close to the grid filter. The LMMSE estimator is not able to estimate the length, because the measurements are uncorrelated with the length. It is interesting to note that all estimators provide the same estimation error for the position. Even the LMMSE estimator, which estimates a totally wrong length gives the same estimation precision for the position.

[1] As the target is static, i.e., does not move, we can omit the time index k.

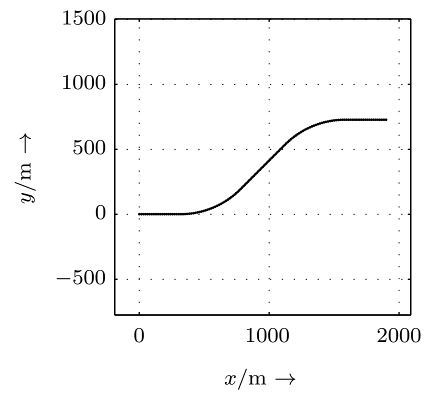


Fig. 6: Trajectory of the stick target.

In Figure 5, the RMSE is depicted for the position and length of the target in scenario 2. The quadratic estimator still yields results very close to the grid filter. The random matrix approach is now not as good as the quadratic estimator, because it does not incorporate the correlation in the prior. The random matrix approach and the LMMSE estimator yield the same RMSE for the center.

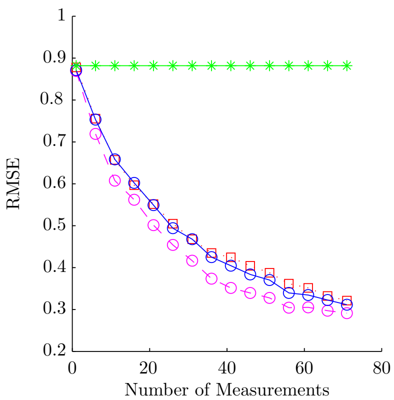
All told, the quadratic estimator yields very precise estimation results close to the optimal Bayes estimator.

B. Tracking a Stick Target

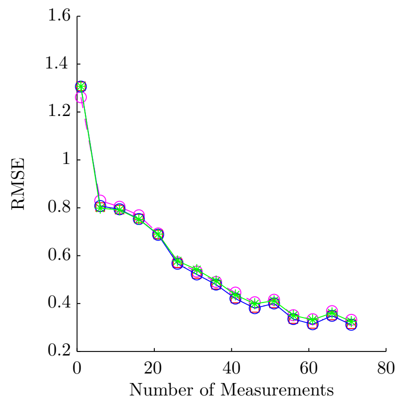
In the following, we show that the quadratic estimator is suitable for tracking a stick target [4]–[6]. In contrast to [4]–[6], where particle filters are used, the quadratic estimator yields a closed-form measurement update. In fact, a stick target can be treated a special case of (6), i.e., the extent in one dimension is known to be zero. The temporal evolution of the target is modeled as a constant velocity model [1]. The trajectory of the stick target is depicted in Figure 6. The stick is aligned with its velocity vector. Figure 7a and Figure 7d show two particular snippets, where the stick target is plotted for several time steps. At each time step, exactly one measurement is received from the stick target. The covariance matrix of the measurement noise is $\mathrm{diag}([0.22, 0.22])$, i.e., rather large in comparison to the length of the stick. These measurements are shown in Figure 7b and Figure 7e. The estimation results of the quadratic estimator are depicted in Figure 7c and Figure 7f. It can be seen that the quadratic estimator yields very precise results. Even though only one measurement per time step is received, the length and orientation of the target is estimated very well.

VI. Conclusions and Future Work

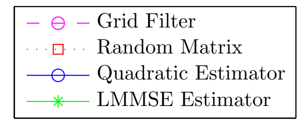
The simultaneous tracking and shape estimation of a target is a high-dimensional nonlinear estimation problem. In this work, we have shown how the problem can be formulated



(a) RMSE for the length.

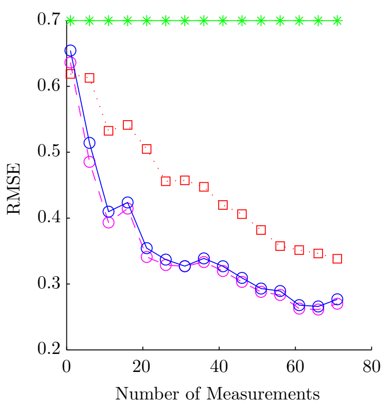


(b) RMSE for the position.

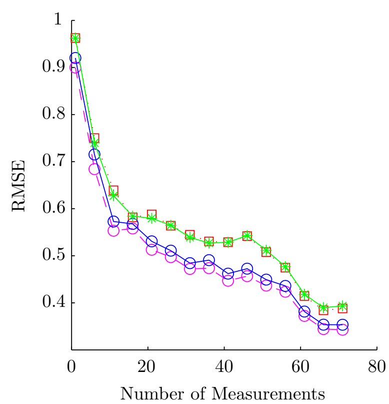


(c) Legend.

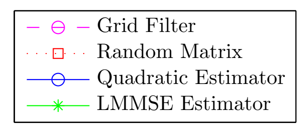
Fig. 4: Estimation results for scenario 1.



(a) RMSE for the length.



(b) RMSE for the position.



(c) Legend.

Fig. 5: Estimation results for scenario 2.

as a measurement equation with multiplicative noise. Based on this equation, we have developed a quadratic estimator for the extent and center of a target. This approach can directly be used for tracking, e.g., elliptic shapes, circular shapes, and stick targets. Due to the Gaussian representation of the uncertainties, this approach lends itself to be embedded into multi-target tracking algorithms such as [8], [9], [26], [27].

Future work will be focused on extending the approach for estimating that parameters of more complex target shapes. This can be achieved by using polynomial measurement noise instead of multiplicative noise.

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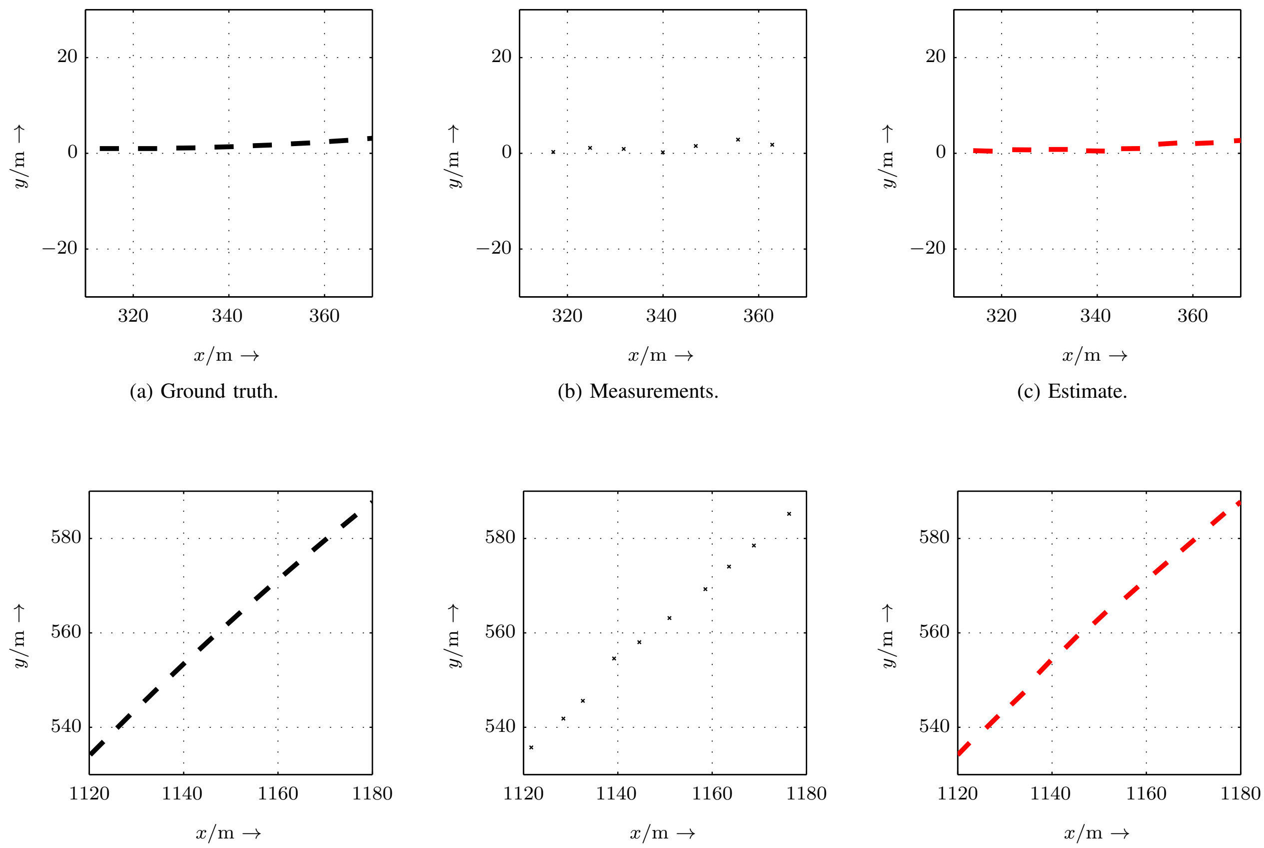

(a) Ground truth. (b) Measurements. (c) Estimate.

(d) Ground truth. (e) Measurements. (f) Estimate.

Fig. 7: Tracking a stick target.

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