Performance Ranking of Multiple Nonlinear Filters Using Ranking Vector and Voting Fusion

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***Abstract*—A lot of performance evaluation metrics exist for nonlinear filters. At present, the most commonly used one is a single and incomprehensive metric of performance. This metric can continuously and quantitatively describe the performance of the nonlinear filters. But in many cases, we need to rank the performance of the filters. It is in general very hard to rank the filters just using a single metric. First, the rankings using a single metric at different times may be different. Then how to get a unique rank for all times? A typical existing solution is to average the single metric over all times. But it is easy to be dominated just by very large values at just some times. Second, a single metric is usually incomprehensive in measuring performance. To make the ranking more comprehensive, multiple metrics are usually needed. But how to get a comprehensive unique rank from the ranks, possibly conflicting with each other? In this paper, we propose a framework to rank multiple nonlinear filters using ranking vectors and voting fusion based on a single metric or multiple metrics. Illustrative examples show that this framework is very effective.**

***Index Terms*—Nonlinear filters, performance ranking, ranking vectors, voting fusion, single metric, multiple metrics**

I. INTRODUCTION

Filtering is a technique using an estimation criterion, through the system model, to estimate state and parameters with noisy measurements. In 1960, R. E. Kalman put forward the Kalman filtering (KF) [1] marked the foundation of modern filtering theory. However, the Kalman filter requires the system to be linear. But in many cases, the models are nonlinear, so we cannot directly apply the Kalman filter. The Extended Kalman filter (EKF) [2] was proposed for nonlinear systems. The EKF approximates nonlinear systems as linear systems by a first-order Taylor series expansion of the nonlinear dynamics equations or measurement equations. Then it uses the standard Kalman filter for state estimation. M. Norgaard proposed divided difference Kalman Filter (DDF) [3] using Stirling interpolation formula. According to the interpolation number, DDFs can be divided into first-order Divided Difference filters (DD1) and second-order Divided Difference filters (DD2). Both EKF and DDF aim at nonlinear function approximation. Julier and Uhlmann put forward the Unscented Kalman filter (UKF) [4], [5] based on the unscented transform to approximate the first two moments used in LMMSE estimation. Another suboptimal filter called Quadrature Kalman filter (QKF) [6], [7] was also proposed. It calculates the first two moments based on Gauss-Hermite integration rules under the hypothesis of Gaussian distributions. However the number of quadrature points of this method grows exponentially with the increase of state dimension. I. Arasaratnam and S. Haykin proposed the Cubature Kalman filter (CKF) in [8] using a third-degree spherical-radial cubature rule to calculate the first two moments. The CKF does not entail any free parameter [9] while the UKF introduces a nonzero scaling parameter to define the nonzero center point. Another well-known nonlinear filter is the particle filter using the sequential Monte Carlo method [10]. This method approximates the discrete posterior density using some random sample points and their corresponding weights. This is different from the above nonlinear filters that just intend to obtain the first two moments of the posterior density.

From the above we can see that much has been done regarding filtering algorithm development. Much less has been done concerning filtering performance evaluation. However, evaluation is as important as filtering algorithm development. It should get more and enough attention.

A comprehensive survey of estimation performance evaluation metrics, e.g., the root mean square error (RMSE), average Euclidean error (AEE), geometric average error (GAE), harmonic average error (HAE), median error (ME), and error mode (EM), was given in [11]. It also points out that the most commonly used RMSE has two serious flaws. First, it is highly large-error dominant. Second, RMSE has no clear physical interpretation. [12] proposes a new measure of central tendency called iterative mid-range error (IMRE). Compared with ME, IMRE is more reliable as it is insensitive to extreme values or the middle value. [13] discusses the drawbacks of existing filter credibility evaluation measures such as normalized estimation error squared (NEES) and average normalized estimation error squared (ANEES), then put forward a new filter credibility tests for bias alone, MSE alone, and joint bias and MSE. The above filter credibility evaluation metrics can be regarded as qualitative analysis. [14] proposes new filter credibility metrics such as noncredibility

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index (NI) and inclination indicators (II) that can measure how credible various self-assessments are.

Furthermore, evaluation methods based on multiple metrics have always been an important part of modern evaluation theory. In fact, most of them are based on weighted averages (see, e.g., [15]-[17]). That is, they combine multiple attributes into a single one for evaluation. There is some controversy in weighted average methods. First, we need to normalize different attributes into consistent units and orders of magnitude. However, there does not exist common methods for this. Second, this method only uses information of each estimator, but not the joint information between two estimators. [18] proposed a new approach, called estimator ranking vector to overcome these drawbacks. This approach uses the joint information like Pitman's closeness measure (PCM) to rank the performance of estimators. [19] presents a metric of performance called error spectrum (ES) which aggregates many incomprehensive metrics. It includes RMSE, AEE, HAE, and GAE as special cases. However, when being applied to dynamic systems, ES will have three dimensions over the total time span, which is not intuitive and difficult to analyze. To overcome this drawback, the dynamic error spectrum (DES) was proposed in [20].

Given multiple filters, most may select a single performance evaluation metric like the frequently used RMSE to determine their ranking. However, ranking filters in terms of the same and single error metric is not without controversy. First, the rankings using a single metric at different times may be different. It is hard to clearly distinguish the performance of the filters at all times. Second, these incomprehensive methods focus on the estimation performance of a certain aspect, so they are one-sided. Furthermore the rankings according to different error metrics may be different. Even when ranking two estimators, there may be given opposite results. In other words, we are not sure which order is reasonable. So, we want to propose a framework for simultaneously solving dynamic system of filtering and solving comprehensive evaluation using multiple metrics to get a final unique ranking of filters.

To rank nonlinear filters using multiple performance metrics, it is better to avoid data normalization and use the joint information between two filters. In this paper, we use the ranking vector to get the filters ranking with both ordering and quantitative information at each time. In many decision making situations, it is necessary to achieve the group consensus. Voting theory [21] can get group consensus well. We make use of voting fusion to obtain the final comprehensive ranking of all filters. Each time can be viewed as a voter, ranking at each time of the dynamic process can be viewed as its decision, and filters can be viewed as candidates. Because the basic idea of voting theory is fairly diverse, the method by which those votes are combined to determine a winner can vary. Different voting methods may give different rankings. We just picked some strategies from voting theory to rank nonlinear filters and present different voting results with or without quantitative information.

The rest of this paper is organized as follows. Section II states the problem. Section III discusses how to rank using the ranking vector. In Section IV, for nonlinear filters of dynamic process, we use voting fusion to obtain the final unique ranking. In Section V, an example is provided to illustrate how to rank the nonlinear filters with multiple performance metrics based on voting fusion. Section VI concludes this paper.

II. Problem Formulation

Consider the following typical nonlinear discrete-time dynamic system

$$\mathbf{x}_{k+1} = f_k(\mathbf{x}_k) + \mathbf{w}_k$$

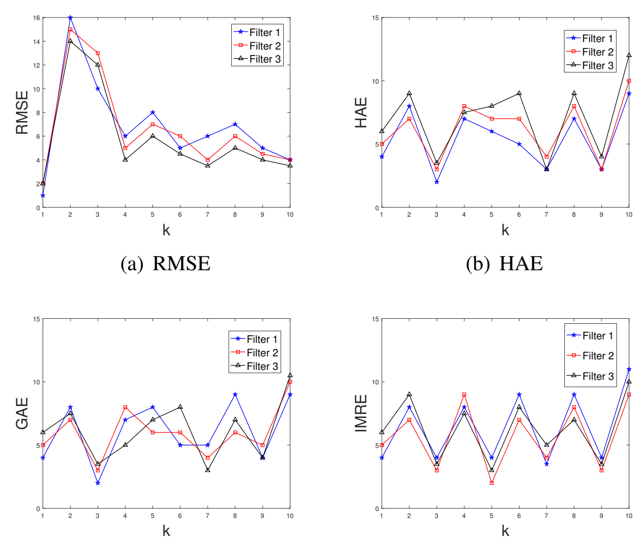
and its corresponding nonlinear measurement model

$$\mathbf{z}_k = h_k(\mathbf{x}_k) + \mathbf{v}_k \ ,$$

where $\mathbf{w}_k$ and $\mathbf{v}_k$ are both zero-mean white Gaussian noise with covariance Q_k and R_k, and uncorrelated with each other.

It is assumed that we have N nonlinear filters and M metrics that depict different aspects of filtering performance. Denote $\hat{x}_k^i = g_i(z^k)$ as the state estimate of filter i at time k, where $z^k = \{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ is the measurement sequence. Our goal is to give a performance ranking of nonlinear filters based on M metrics.

Consider the following example in which there are three nonlinear filters dealing with noisy data to obtain the state estimation. We use four different attribute measures, i.e., RMSE, HAE, GAE, IMRE, to evaluate the performance of filters as shown in Fig. 1.



(a) RMSE (b) HAE

(c) GAE (d) IMRE

Fig. 1. Performance Evaluation of Filters

First, from Fig. 1(a), we can see that the ranking is $\hat{x}_2^3 \succ \hat{x}_2^2 \succ \hat{x}_2^1$ at time $k = 2$, but $\hat{x}_6^3 \succ \hat{x}_6^1 \succ \hat{x}_6^2$ at time $k = 6$, where "$\succ$" means that the performance of the filter in the left is better than that of the filter in the right. The rankings at different times are different. Furthermore, from Fig. 1(c) we can see that the GAE of three filters are interleaved with each other. Then the question is how to get the final unique ranking just based on a single metric, e.g., GAE, over all times. Second, the rankings may be inconsistent according to four different performance metrics as in Fig.1 (a)-(d). How to overcome this inconsistency? How to get a unique ranking of multiple nonlinear filters based on multiple metrics?

III. First Ranking using Ranking Vectors

A. Ranking Vectors

Our main goal is to rank nonlinear filters, but at the same time it is better to also have a quantitative reference. Next we introduce the concept of a ranking vector [18] that can provide both ordering and quantitative information. Define a ranking vector as

$$r_k = \left[r_k^1, r_k^2, \cdots, r_k^N\right]^T ,$$

where r_k^i is the value of filter i at time k and $r_k^i > 0$ and N is the number of filters. The elements of a ranking vector are all positive and stand for the strengths of the filters relative to each other. The larger the element, the better the corresponding filter. Therefore, the ranking vector elements reflect the order of the filters.

B. PCM Matrix

To compare the performance of parameter estimators, Pitman [22] proposed a criterion based on the probability or frequency that one estimator is closer to the truth than the other. If this probability is larger than 0.5, then the first estimator can be considered to be superior to the second. This criterion is known as Pitman's closeness measure (PCM) [23]. The definition of PCM is given in the following. Given an estimated parameter x and its estimator $\hat{x}$, let $m(1,2)$ denote the measure of the difference between two estimators $\hat{x}^1$ and $\hat{x}^2$ relative to the parameter x

$$m(1,2) \triangleq \begin{cases} 1 & \text{if } \hat{x}^1 \succ \hat{x}^2 \\ 0.5 & \text{if } \hat{x}^1 = \hat{x}^2 \\ 0 & \text{if } \hat{x}^1 \prec \hat{x}^2 \end{cases} , \quad (1)$$

where $\hat{x}^1 \succ \hat{x}^2$ means that $\hat{x}^1$ is closer to the truth than $\hat{x}^2$.

Let $\text{PCM}(1,2)$ denote PCM between two estimators $\hat{x}^1$ and $\hat{x}_2$ relative to the parameter x

$$\begin{aligned} \text{PCM}(1,2) &= E\left[m(1,2)\right] \\ &= \Pr\left\{\hat{x}^1 \succ \hat{x}^2\right\} + 0.5\Pr\left\{\hat{x}^1 = \hat{x}^2\right\} , \end{aligned}$$

If $\text{PCM}(1,2) > 0.5$, we can consider that $\hat{x}^1$ is closer to the truth than $\hat{x}^2$ in the sense of Pitman's closeness. According to the definition of PCM, we can see that PCM just describes the joint information between two estimators. Due to the drawbacks of nontransitivity described in [24], a competition matrix called PCM matrix needs to be introduced

$$C_{\text{PCM}} = \begin{bmatrix} \text{PCM}(1,1) & \cdots & \text{PCM}(1,N) \\ \vdots & \ddots & \vdots \\ \text{PCM}(N,1) & \cdots & \text{PCM}(N,N) \end{bmatrix} . \quad (2)$$

It can be seen that the PCM matrix contains the entire pairwise competitions of all estimators based on PCM.

C. Ranking Method Based on a Single Metric

After getting the PCM matrix, we need to specify a method to get the ranking vector. Then we can get the filters' order.

1) Sum Score Vector of the First Round (SSV1): One simple method is to calculate the sum score vector of the first round, defined as

$$\text{SSV1} = C_{\text{PCM}} \cdot \mathbf{1}_{N\times 1} , \quad (3)$$

where $\mathbf{1}_{N\times 1}$ is a column vector with all elements equal to one.

SSV1 ignores the prior strength (e.g., all equal to 1) of the filters and treats them without difference [25]. Then its ith element represents the sum of the ith filter's Pitman's closeness measure when competing with all other filters.

2) Sum Score Vector of the Second Round (SSV2): Since SSV1 does not make full use of joint information provided by the competition matrix, an improved method called sum score vector of the second round was proposed. The definition of SSV2 is

$$\text{SSV2} = C_{\text{PCM}}^2 \cdot \mathbf{1}_{N\times 1} = C_{\text{PCM}} \cdot \text{SSV1} . \quad (4)$$

SSV2 treats SSV1 as the prior information. By pre-multiplying the PCM matrix once more, we can explore more joint information from the PCM matrix.

3) Order-Preserving Mapping (OPM): First we define the quality vector $q = \left[q_1, \cdots, q_N\right]^T$, where q_i is the quality of the ith filter and $q_i > 0$. Using the PCM matrix obtained from the above, we assume $r = C_{\text{PCM}} \cdot q$. In order to ensure that the ranking vector r and the quality vector q have the same order, we assume that they have a linear relationship as $r = \lambda q$. Now, if we can get q, we can get the ranking vector r equivalently [18].

In view of the above, let

$$C_{\text{PCM}} \cdot q = \lambda q . \quad (5)$$

Now, obtaining the ranking vector becomes finding eigenvectors of PCM matrix. According to Perron-Frobenius theorem [26], if we have a positive PCM, then we can obtain the only eigenvector with all elements to be positive. If there are zero elements in C_{PCM}, they can be simply replaced by very small positive numbers, e.g., 0.001. Thus, the effect of substitution will be alleviated.

D. Ranking Method Based on Multiple Attributes

The above ranking methods are just for a single metric, but not for multiple metrics. Because a single metric is in general incomprehensive, it is preferable to use multiple metrics. Next, we define multiple metrics competition measure (MCM) to fully use comprehensive joint information between two filters.

Let $m_{\text{MCM}}(1,2;a_i)$ denote the PCM of filters $\hat{x}^1$ and $\hat{x}^2$ relative to the ith metric a_i

$$m_{\text{MCM}}(1,2;a_i) \triangleq \begin{cases} 1 & \text{if } \hat{x}^1 \succ \hat{x}^2 \\ 0.5 & \text{if } \hat{x}^1 = \hat{x}^2 \\ 0 & \text{if } \hat{x}^1 \prec \hat{x}^2 \end{cases} . \quad (6)$$

Then, the MCM is defined as

$$\text{MCM}(1,2) = \frac{1}{M}\sum_{i=1}^{M} m_{\text{MCM}}(1,2;a_i)\ . \tag{7}$$

When $\text{MCM}(1,2) > 0.5$, we can consider that $\hat{x}^1$ is better than $\hat{x}^2$ in the sense of multiple metrics competition. Similar to the PCM, MCM also just describes the joint information between two filters. So we define MCM matrix to include the entire pairwise competition information of all filters

$$C_{\text{MCM}} = \begin{bmatrix} \text{MCM}(1,1) & \cdots & \text{MCM}(1,N) \\ \vdots & \ddots & \vdots \\ \text{MCM}(N,1) & \cdots & \text{MCM}(N,N) \end{bmatrix}\ . \tag{8}$$

After obtaining the MCM matrix, we can use the above ranking method like SSV1, SSV2 and Order-Preserving Mapping to get the ranking vector. Then we can rank the nonlinear filters based on multiple metrics.

IV. Second Ranking using Voting Fusion

Nonlinear filtering is used for dynamic processes. However, the above ranking methods can only be applied at a specific time step. As a result of this, we can get the filters ranking vectors at every time. How to get a final unique order for all nonlinear filters? Next we provide some ranking fusion methods to solve this problem.

A. Summation Method

The main idea is that ranking vectors of all times are summed up and then normalized to get a final order. First if a filter is better than another one, it shows that the performance of this filter is better than another one in most of time. Second, ranking vector contains the information of quantity, so it can quantitatively tell us how big the difference between the filters at each time. This method counts the difference of all times and has certain rationality. The combined ranking vector is defined as

$$\bar{r} = \sum_{i=1}^{k} r_i \Bigg/ \left\| \sum_{i=1}^{k} r_i \right\|_2\ , \tag{9}$$

where r_i is the ranking vector at the ith time, and $\bar{r}$ is the combined ranking vector.

The disadvantage of this method is that the ranking is easily corrupted by extreme values. For example, at a time, if the ranking vector element of one filter is extremely large, it will dominate the final ranking vector.

Example 1. Suppose that we have the ranking vectors of 3 filters at 3 times

$$r = \begin{bmatrix} r_1^1 & r_2^1 & r_3^1 \\ r_1^2 & r_2^2 & r_3^2 \\ r_1^3 & r_2^3 & r_3^3 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 & 0.5 \\ 0.4 & 2.1 & 0.3 \\ 0.3 & 0.2 & 0.2 \end{bmatrix}\ , \tag{10}$$

where r_j^i represents the ranking vector component of filter i at time j, and r is matrix of ranking vectors at all times.

Then $\bar{r} = \sum_{i=1}^{k} r_i \Big/ \left\| \sum_{i=1}^{k} r_i \right\|_2 = [\ 0.2708 \quad 0.5833 \quad 0.1458\ ]^T$, according to the summation method. The final ranking is $\hat{x}^2 \succ \hat{x}^1 \succ \hat{x}^3$. But we can see that the ranking at both time 1 and 3 is $\hat{x}^1 \succ \hat{x}^2 \succ \hat{x}^3$. In other words, the ranking is $\hat{x}^1 \succ \hat{x}^2 \succ \hat{x}^3$ at most times. Just because of one time of dominant ranking, the final order is totally changed.

B. Plurality Method

Majority Criterion [28]: If a choice has a majority of first-place votes, that choice should be the winner.

This method is known to us as majority criterion in voting theory, namely the candidate with most votes is declared the winner. We first sort every times ranking vector $r_k = \left[r_k^1, r_k^2, \cdots, r_k^N\right]^T$ in descending order $r_k = [r_k^{(1)}, r_k^{(2)}, \cdots, r_k^{(N)}]^T$. Then we use the ranking vectors in descending order at all times to define a preference matrix as

$$R = \begin{bmatrix} r_1^{(1)} & \cdots & r_k^{(1)} \\ \vdots & \ddots & \vdots \\ r_1^{(N)} & \cdots & r_k^{(N)} \end{bmatrix}\ , \tag{11}$$

where k denotes time and N denotes the number of filters.

From the first row of the preference matrix, we select the filter with the largest number as the best according to the plurality principle. We remove the first filter from the preference matrix and it will become a dimension-reduced preference matrix, each column remaining in a descending order. Then we select the filter with the largest number from the first row of the dimension reduction preference matrix as the second best, and so on.

The advantage is that the ranking obtained by plurality method is not corrupted by extreme values. The disadvantage of this method is that it can only provide order information but not quantitative information. If two or more filters have the same largest number in the first row, then this method will fail.

Example 2. Suppose that we have the ranking vectors of 3 filters at 4 times

$$r = \begin{bmatrix} r_1^1 & r_2^1 & r_3^1 & r_4^1 \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 \\ r_1^3 & r_2^3 & r_3^3 & r_4^3 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 & 0.5 & 0.6 \\ 0.4 & 2.1 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.2 & 0.5 \end{bmatrix}\ .$$

Then the preference matrix is

$$\begin{aligned} R &= \begin{bmatrix} r_1^{(1)} & r_1^{(1)} & r_1^{(1)} & r_4^{(1)} \\ r_1^{(2)} & r_2^{(2)} & r_3^{(2)} & r_4^{(2)} \\ r_1^{(3)} & r_2^{(3)} & r_3^{(3)} & r_4^{(3)} \end{bmatrix} = \begin{bmatrix} 0.5 & 2.1 & 0.5 & 0.6 \\ 0.4 & 0.3 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.2 & 0.4 \end{bmatrix} \\ &= \begin{bmatrix} r_1^1 & r_2^2 & r_3^1 & r_4^1 \\ r_1^2 & r_2^1 & r_3^2 & r_4^3 \\ r_1^3 & r_2^3 & r_3^3 & r_4^2 \end{bmatrix}\ . \end{aligned}$$

Next we can see that filter 1 has the largest number in the first row of the above matrix. So we select filter 1 as the first best and remove filter 1 from the preference matrix. Now we can get the reduced preference matrix

$$R = \begin{bmatrix} 0.4 & 2.1 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} r_1^2 & r_2^2 & r_3^2 & r_4^3 \\ r_1^3 & r_2^3 & r_3^3 & r_4^2 \end{bmatrix}\ .$$

Then we select filter 2 as the second best because filter 2 has the largest number in first row of reduced preference matrix.

Then we repeat the above steps until the ranking is finished. The final ranking of plurality method is $\hat{x}^1 \succ \hat{x}^2 \succ \hat{x}^3$. At the same time, we get the ranking is $\hat{x}^2 \succ \hat{x}^1 \succ \hat{x}^3$ by summation method. So we can see that the final ranking of this method is not affected by the extreme value at time 2.

Condorcet Criterion [27]: If there is a choice that is preferred in every one-to-one comparison with the other choices, that choice should be the winner. We call this winner the Condorcet Winner.

Example 3. Suppose that we have the ranking vectors of 3 filters at 7 times

$$\begin{aligned} r &= \begin{bmatrix} r_1^1 & r_2^1 & r_3^1 & r_4^1 & r_5^1 & r_6^1 & r_7^1 \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 & r_5^2 & r_6^2 & r_7^2 \\ r_1^3 & r_2^3 & r_3^3 & r_4^3 & r_5^3 & r_6^3 & r_7^3 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 & 0.8 & 0.5 & 0.4 & 0.5 & 0.4 & 0.4 \\ 0.4 & 0.2 & 0.3 & 0.6 & 0.4 & 0.3 & 0.6 \\ 0.3 & 0.5 & 0.4 & 0.5 & 0.7 & 0.6 & 0.5 \end{bmatrix}. \end{aligned}$$

Then the preference matrix is

$$\begin{aligned} R &= \begin{bmatrix} 0.5 & 0.8 & 0.5 & 0.6 & 0.7 & 0.6 & 0.6 \\ 0.4 & 0.5 & 0.4 & 0.5 & 0.5 & 0.4 & 0.5 \\ 0.3 & 0.2 & 0.3 & 0.4 & 0.4 & 0.3 & 0.4 \end{bmatrix} \qquad (12) \\ &= \begin{bmatrix} r_1^1 & r_2^1 & r_3^1 & r_4^2 & r_5^3 & r_6^3 & r_7^2 \\ r_1^2 & r_2^3 & r_3^3 & r_4^3 & r_5^1 & r_6^1 & r_7^3 \\ r_1^3 & r_2^2 & r_3^2 & r_4^1 & r_5^2 & r_6^2 & r_7^1 \end{bmatrix}. \end{aligned}$$

According to the plurality method, we can get the order of filters as $\hat{x}^1 \succ \hat{x}^3 \succ \hat{x}^2$. The ranking above may seem valid, but there is a problem. Looking back at our preference matrix, 5 out of the 7 times ($k = 1, 2, 3, 5, 6$) we prefer filter 1 to filter 2. And 4 out of the 7 times ($k = 4, 5, 6, 7$) we prefer filter 3 to filter 1. Also we see that filter 3 is preferred to filter 2. Finally, filter 3 is the Condorcet winner. So this method may violate the Condorcet Criterion.

C. Instant Runoff Method

The Instant Runoff Method (IRM), also called Plurality with Elimination, is a modification of the plurality method. The main idea is to use the runoff step in which the filter with the least number in the first row of the preference matrix is eliminated. The dimension of the preference matrix is reduced for the next elimination and the descending order of each column is still remained. We then repeat the above steps until the first best filter is obtained. Similarly, we runoff the first filter in the original preference matrix and repeat runoff steps to find the second filter.

The advantage is that the ranking obtained by instant runoff method is not corrupted by extreme values. The disadvantage of this method is still that it can only provide an order of the filters but without quantitative information about them.

Example 4. Suppose that we have the same case as in example 2.

Round 1: From the preference matrix R we can see that filter 3 has the least number in the first row, so we eliminate it. Then we have

$$R = \begin{bmatrix} 0.5 & 2.1 & 0.5 & 0.6 \\ 0.4 & 0.3 & 0.3 & 0.4 \end{bmatrix} = \begin{bmatrix} r_1^1 & r_2^2 & r_3^1 & r_4^1 \\ r_1^2 & r_2^1 & r_3^2 & r_4^2 \end{bmatrix}$$

Round 2: From the above preference matrix R we can see that filter 2 has the least number in the first row, so we eliminate it. Then we can get:

$$R = \begin{bmatrix} 0.5 & 0.3 & 0.5 & 0.6 \end{bmatrix} = \begin{bmatrix} r_1^1 & r_2^1 & r_3^1 & r_4^1 \end{bmatrix}$$

Round 3: we select filter 1 as the first best filter. Next we runoff filter 1 in the original preference matrix. Then we can get:

$$R = \begin{bmatrix} 0.4 & 2.1 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} r_1^2 & r_2^2 & r_3^2 & r_4^3 \\ r_1^3 & r_2^3 & r_3^3 & r_4^2 \end{bmatrix}$$

Repeat runoff steps until the ranking is obtained.

Example 5. Suppose that we have the same case as in example 3.

According to the instant runoff method, we can get the order of the filters as $\hat{x}^1 \succ \hat{x}^3 \succ \hat{x}^2$. Similarly, this method may violate the Condorcet Criterion, too.

D. Borda Count Method

The Borda count principle gives each filter in the descending order at each time a numerical value. Assuming that the descending order of the filters at time k is $r_k = [r_k^{(1)}, r_k^{(2)}, \cdots, r_k^{(N)}]^T$, we assign the corresponding values as $[N, N-1, \cdots, 1]^T$. Then the numerical values of all filters at all times are summed up. Finally, we normalize the sum vector to get the final ranking vector.

The advantage is that the ranking obtained by the Borda count method is not corrupted by extreme values. At the same time this method overcomes the drawback of missing quantitative information in the plurality method. But if there are filters of the same performance, it is not easy to assign values to them.

Example 6. Suppose that we have the same case as in example 1.

Then from (10) we can get the corresponding preference matrix

$$R = \begin{bmatrix} 0.5 & 2.1 & 0.5 \\ 0.4 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.2 \end{bmatrix} = \begin{bmatrix} r_1^1 & r_2^2 & r_3^1 \\ r_1^2 & r_2^1 & r_3^2 \\ r_1^3 & r_2^3 & r_3^3 \end{bmatrix}.$$

If we assign corresponding values of ranking vector in descending order as $[3, 2, 1]^T$.

The scores of the filters are summed up as in Table I.

Table I: Scores of filters for example 6

Filter	1	2	3
Score	8	7	3

Next we normalize the scores to get the final ranking vector as $r = \begin{bmatrix} 0.4444 & 0.3899 & 0.1667 \end{bmatrix}^T$. The final ranking is $\hat{x}^1 \succ \hat{x}^2 \succ \hat{x}^3$. This is different from the rank of the summation method and overcomes its drawback.

Example 7. Suppose that we have the same case as in example 3. We assign corresponding values of ranking vector in descending order as $[3, 2, 1]^T$. From (12) , the scores of the filters are as in Table II.

Table II: Scores of filters for example 7

Filter	1	2	3
Score	15	12	15

According to the Borda count method, we can get the final ranking vector as $r = \begin{bmatrix} 0.3571 & 0.2857 & 0.3571 \end{bmatrix}^T$. Then the final ranking is $\hat{x}^1 = \hat{x}^3 \succ \hat{x}^2$. From the result of example 3, we can see that filter 1 is the majority one. So filter 1 should be the best one. However, it is obvious that Borda Count method may violate the Majority Criterion.

E. Pairwise Comparison Method

In this method, each pair of filters is compared, using ranking vectors at all times to determine which of the two is preferred. The preferred filter is awarded 1 point. If there is a tie, each filter is awarded 0.5 point. After all pairwise comparisons are made at each time, the values of each filter are obtained. The score is defined as,

$$u_k^i(i,j) = \begin{cases} 1 & \text{if } \hat{x}_k^i \succ \hat{x}_k^j \\ 0.5 & \text{if } \hat{x}_k^i = \hat{x}_k^j \\ 0 & \text{if } \hat{x}_k^i \prec \hat{x}_k^j \end{cases} , \tag{13}$$

where $u_k^i(i,j)$ denotes the point of filter i obtained by its comparison with filter j at time k. $\hat{x}_k^i \succ \hat{x}_k^j$ means that the performance of filter i is better than that of filter j at time k.

Next, each filter adds its own score of all times to get its own final score. Finally, we use normalization to get the final unique ranking vector.

The advantage is that the ranking obtained by pairwise comparisons method is not corrupted by extreme values. At the same time this method overcomes the drawback of without quantitative information in the plurality method and difficult to assign values in the borda count method. Because this method is specifically designed to satisfy the Condorcet Criterion by looking at pairwise (one-to-one) comparisons, it obviously satisfies the Condorcet Criterion.

Example 8. Suppose that we have the ranking vectors of 3 filters at 4 times

$$r = \begin{bmatrix} r_1^1 & r_2^1 & r_3^1 & r_4^1 \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 \\ r_1^3 & r_2^3 & r_3^3 & r_4^3 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.2 & 0.5 \\ 0.3 & 0.2 & 0.4 & 0.6 \\ 0.4 & 0.4 & 0.5 & 0.4 \end{bmatrix} .$$

Then the preference matrix is

$$R = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.6 \\ 0.4 & 0.4 & 0.4 & 0.5 \\ 0.3 & 0.2 & 0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} r_1^1 & r_2^1 & r_3^3 & r_4^2 \\ r_1^3 & r_2^3 & r_3^2 & r_4^1 \\ r_1^2 & r_2^2 & r_3^1 & r_4^3 \end{bmatrix} . \tag{14}$$

From (14) we can see that filter 1 is better than filter 2 at time 1 and 2, but worse than filter 2 at time 3 and 4. So filter 1 gets 2 points and filter 2 gets 2 points. Then we can see that filter 1 is better than filter 3 at time 1, 2, and 4. So filter 1 gets 3 points and filter 3 gets 1 points. Similarly, we can get the following scores when comparing with each other in Table III.

Table III: Pairwise comparison of filters

Points	Filter 1	Filter 2	Filter 3
Filter 1	0	2	3
Filter 2	2	0	1
Filter 3	1	3	0

In Table III, we just make row comparisons. For example, the first row means that the scores of filter 1 got by comparing with filter 2 and filter 3. Similarly, the second row means the scores of filter 2, and the third row means the scores of filter 3. Finally, the total scores of each filter is in Table IV.

Table IV: Scores of filters for example 8

Filter 1	5
Filter 2	3
Filter 3	4

The final normalized ranking vector is $r = \begin{bmatrix} 0.4167 & 0.2500 & 0.3333 \end{bmatrix}^T$. The final ranking is $\hat{x}^1 \succ \hat{x}^3 \succ \hat{x}^2$.

The Independence of Irrelevant Alternatives (IIA) Criterion [28]: If a non-winning choice is removed from the ballot, it should not change the winner of the election.

Example 9. Suppose that we have the same case as in example 8

If we remove filter 3, the preference matrix is

$$R = \begin{bmatrix} 0.5 & 0.5 & 0.4 & 0.6 \\ 0.3 & 0.2 & 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} r_1^1 & r_2^1 & r_3^2 & r_4^2 \\ r_1^2 & r_2^2 & r_3^1 & r_4^1 \end{bmatrix}$$

Then the total points is as in Table V,

Table V: Scores of filters for example 9

Filter 1	2
Filter 2	2

It can be seen that after we remove the non-winning filter 3, the ranking is changed. In this example, the IIA Criterion was violated. Another disadvantage of Pairwise Comparisons method is that it is fairly easy for ranking to end in a tie.

V. ILLUSTRATIVE EXAMPLES

The following examples demonstrate how the voting fusion can be used for single-metric and multiple-metric ranking.

A. Scalar Dynamic System

To illustrate the effectiveness and rationality of the proposed ranking method for multiple nonlinear filters based on voting fusion, we consider the following scalar nonlinear dynamic system

$$x_k = \frac{1}{2}x_{k-1} + \frac{25x_{k-1}}{1+x_{k-1}^2} + 8\cos[1.2(k-1)] + w_{k-1} ,$$

which is observed through

$$z_k = \frac{1}{20}x_k^2 + v_k .$$

The parameters are

$$x_0 \sim N(0.1, 2), \; w_k \sim N(0,1), \; v_k \sim N(0,1) .$$

This scalar nonlinear system works as a benchmark testing example in many existing work for nonlinear filtering [10].

Next we consider six nonlinear filters such as DD1, DD2, EKF, UKF, CKF, QKF for ranking over 500 Monte-Carlo runs.

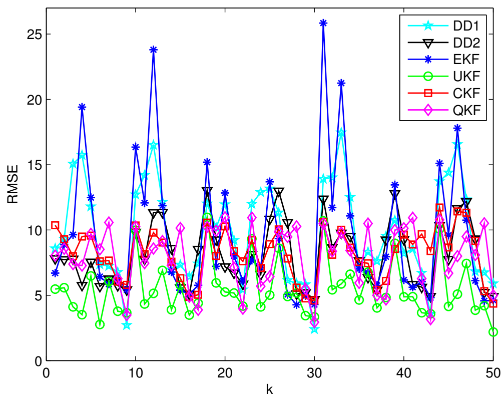


Fig. 2. Filtering RMSE

Table VI: Rankings based on RMSE

Summation Method	
SSV1	DD1≺EKF≺CKF≺DD2≺QKF≺UKF $r = [0.2168, 0.3768, 0.3329, 0.6451, 0.3432, 0.4077]^T$
SSV2	DD1≺CKF≺EKF≺DD2≺QKF≺UKF $r = [0.1396, 0.3199, 0.2858, 0.7642, 0.2610, 0.3800]^T$
OPM	DD1≺CKF≺EKF≺DD2≺QKF≺UKF $r = [0.1148, 0.2505, 0.2453, 0.8479, 0.1882, 0.3309]^T$
Plurality Method	
SSV1	DD1≺EKF≺CKF≺DD2≺QKF≺UKF
SSV2	DD1≺EKF≺CKF≺DD2≺QKF≺UKF
OPM	DD1≺EKF≺CKF≺DD2≺QKF≺UKF
Instant Runoff Method	
SSV1	DD1≺EKF≺CKF≺DD2≺QKF≺UKF
SSV2	DD1≺EKF≺CKF≺DD2≺QKF≺UKF
OPM	DD1≺EKF≺CKF≺DD2≺QKF≺UKF
Borda Count Method	
SSV1	DD1≺EKF≺CKF≺DD2≺QKF≺UKF $r = [0.2444, 0.3834, 0.3453, 0.6166, 0.3543, 0.4103]^T$
SSV2	DD1≺EKF≺CKF≺DD2≺QKF≺UKF $r = [0.2444, 0.3834, 0.3453, 0.6166, 0.3543, 0.4103]^T$
OPM	DD1≺EKF≺CKF≺DD2≺QKF≺UKF $r = [0.2444, 0.3834, 0.3453, 0.6166, 0.3543, 0.4103]^T$
Pairwise Comparisons Method	
SSV1	DD1≺EKF≺CKF≺DD2≺QKF≺UKF $r = [0.2444, 0.3834, 0.3453, 0.6166, 0.3543, 0.4103]^T$
SSV2	DD1≺EKF≺CKF≺DD2≺QKF≺UKF $r = [0.2444, 0.3834, 0.3453, 0.6166, 0.3543, 0.4103]^T$
OPM	DD1≺EKF≺CKF≺DD2≺QKF≺UKF $r = [0.2444, 0.3834, 0.3453, 0.6166, 0.3543, 0.4103]^T$

First of all, we use a single metric such as the RMSE for ranking. Fig. 2 shows the RMSE of six filters. Table VI shows the ranking based on RMSE. From Fig. 2, we can see that the worst is EKF. However, we can see that the ranking of EKF is not the worst from Table VI. We can see from Fig. 2 that DD1 is actually worse than EKF at most of time and better than EKF only at some times. It illustrates that a few large values in single metric ranking do not influence the final order. We can see that the order of the filters using summation method is not the same. Because different ranking vector methods are based on different criteria and different fusion methods are based on different criteria, the final order may be different. For example, SSV1 and SSV2 have difference in prior information. This will lead to different order. Now, we evaluate performance of multiple nonlinear filters based on multiple metrics. First, we get multi-metric performance of filters at a specific time, e.g., $k = 50$.

Table VII: Multi-metric performance of filters at time $k = 50$

	DD1	DD2	EKF	UKF	CKF	QKF
RMSE	5.8959	4.9097	4.6247	2.1942	4.3652	4.9983
AEE	2.4060	3.7855	1.9264	1.2061	1.9646	4.4303
HAE	0.1782	0.9590	0.1768	0.0364	0.1698	3.5695
GAE	0.7247	2.9573	0.6660	0.6494	0.7801	3.9713
ME	0.7879	3.2213	0.7056	0.7693	0.8718	4.0205
EM	0.2571	3.0245	0.4400	0.4132	0.4476	3.8485
IMRE	1.2342	3.4425	1.0602	0.9678	1.2348	4.2437

Six filters and seven metrics are considered in Table VII. Our goal is to rank the comprehensive quality of the six filters. Let filters compete with each other just like PCM does. For example, DD1 is better than DD2 for metrics of AEE, GAE, ME, EM, IMRE. Thus, $m_{\text{MCM}}(1, 2; a_2) = m_{\text{MCM}}(1, 2; a_3) = m_{\text{MCM}}(1, 2; a_4) = m_{\text{MCM}}(1, 2; a_5) = m_{\text{MCM}}(1, 2; a_6) = m_{\text{MCM}}(1, 2; a_7) = 1$. Then, we have

$$\text{MCM}(1, 2) = \frac{1}{7}\sum_{i=1}^{7} m_{\text{MCM}}(1, 2; a_i) = \frac{1}{7} * 6 = 0.8571 \ .$$

Table VIII: Rankings based on voting fusion

Summation Method	
SSV1	QKF≺DD2≺CKF≺DD1≺EKF≺UKF $r = [0.3957, 0.2634, 0.5400, 0.5926, 0.2784, 0.2316]^T$
SSV2	QKF≺DD2≺CKF≺DD1≺EKF≺UKF $r = [0.3657, 0.2079, 0.5739, 0.6377, 0.2319, 0.1822]^T$
OPM	QKF≺DD2≺CKF≺DD1≺EKF≺UKF $r = [0.3382, 0.1818, 0.5894, 0.6603, 0.2059, 0.1634]^T$
Plurality Method	
SSV1	QKF≺DD2≺CKF≺DD1≺EKF≺UKF
SSV2	QKF≺DD2≺CKF≺DD1≺EKF≺UKF
OPM	QKF≺DD2≺CKF≺DD1≺EKF≺UKF
Instant Runoff Method	
SSV1	QKF≺DD2≺CKF≺DD1≺EKF≺UKF
SSV2	QKF≺DD2≺CKF≺DD1≺EKF≺UKF
OPM	QKF≺DD2≺CKF≺DD1≺EKF≺UKF
Borda Count Method	
SSV1	QKF≺DD2≺CKF≺DD1≺EKF≺UKF $r = [0.3828, 0.2516, 0.5513, 0.5929, 0.2713, 0.2472]^T$
SSV2	QKF≺DD2≺CKF≺DD1≺EKF≺UKF $r = [0.4054, 0.2463, 0.5579, 0.5841, 0.2593, 0.2354]^T$
OPM	QKF≺DD2≺CKF≺DD1≺EKF≺UKF $r = [0.4051, 0.439, 0.5597, 0.5837, 0.2592, 0.2352]^T$
Pairwise Comparisons Method	
SSV1	QKF≺DD2≺CKF≺DD1≺EKF≺UKF $r = [0.1707, 0.0853, 0.2707, 0.2920, 0.0973, 0.0840]^T$
SSV2	QKF≺DD2≺CKF≺DD1≺EKF≺UKF $r = [0.1813, 0.0840, 0.2747, 0.2907, 0.0920, 0.0773]^T$
OPM	QKF≺DD2≺CKF≺DD1≺EKF≺UKF $r = [0.1813, 0.0827, 0.2760, 0.2907, 0.0920, 0.0773]^T$

By pairwise competition, the following MCM matrix is obtained

$$C_{\text{MCM}} = \begin{bmatrix} 0.5000 & 0.8571 & 0.1429 & 0.1429 & 0.5714 & 0.8571 \\ 0.1429 & 0.5000 & 0 & 0 & 0 & 1 \\ 0.8571 & 1 & 0.5000 & 0.1429 & 0.0714 & 1 \\ 0.8571 & 1 & 0.8571 & 0.5000 & 1 & 1 \\ 0.4286 & 1 & 0.2857 & 0 & 0.5000 & 1 \\ 0.1429 & 0 & 0 & 0 & 0 & 0.5000 \end{bmatrix} .$$

Then according to the SSV1, SSV2, and OPM methods, we can obtain the ranking vector at time $k=50$ as

$$r_{\text{SSV1}} = \begin{bmatrix} 3.0714 & 1.6429 & 4.2143 & 5.21433 & 3.2143 & 0.6429 \end{bmatrix}^T,$$
$$r_{\text{SSV2}} = \begin{bmatrix} 6.6786 & 1.9031 & 10.0663 & 14.3520 & 6.4133 & 0.7602 \end{bmatrix}^T,$$
$$r_{\text{OPM}} = \begin{bmatrix} 0.3054 & 0.0583 & 0.4862 & 0.7705 & 0.2687 & 0.0335 \end{bmatrix}^T.$$

Similarly, we can get the order at every time. Then we use voting fusion to get the final order.

Different rankings and ranking vectors are shown in Table VIII. We can see that different methods can give different rankings. This is because different methods are based on different criteria. Also, we can see that most of the rankings are the same, e.g., summation method, Borda Count method, and pairwise comparisons method, This illustrates that our framework is reasonable.

VI. Conclusion

Although the idea of the multiple-metric decision method based on weighted average can be used to solve the filter ranking problem, it is not without controversy and has some difficulties when directly applied to filter ranking problem. First, we have to overcome the difficulty of the dynamic system of filtering, which may have different rankings at different times. Second, we need to find a consensus method to normalize the different metrics into consistent units and orders of magnitude. In this paper, we first applied three types of ranking methods based on ranking vectors to obtain the ranking of filters at every time. Then we used methods from voting theory such as the summation method, the plurality method one, the instant runoff method, the Borda count method, and the pairwise comparison method to obtain the final ranking of nonlinear filters based on multiple metrics. We analyzed the pros and cons of different methods. The example of a scalar dynamic system illustrates that our framework is quite effective. In this paper, we just provided a few typical voting fusion strategies. Other strategies can also be used and will be discussed in our future work.

References

[1] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Transactions of the ASME, Journal of Basic Engineering*, vol. 82, no. 1, pp. 35-45, 1960.

[2] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan, *Estimation with applications to tracking and navigation*, New York: John wiley&Sons, Inc, 2001.

[3] M. Norgaard, N. K. Poulsen, and O. Ravn, "New developments in state estimation of nonlinear systems," *Automatica*, vol. 36, no. 11, pp. 1627-1638, 2000.

[4] S. J. Julier, J. K. Uhlmann, and Durrant-Whyte, "A new method for the nonlinear transformation of means and covariances in filters and estimators," *IEEE Transactions on Automatic Control*, vol. 45, no. 3, pp. 472-482, 2000.

[5] S. J. Julier, J. K. Uhlmann, and Durrant-Whyte, "Unscented filtering and nonlinear estimation," *Proceedings of the IEEE*, vol. 92, no. 3, pp. 401-422, 2004.

[6] K. Ito, K. Xiong, "Gaussian filters for nonlinear filtering problems," *IEEE Transactions on Automatic Cotrol*, vol. 45, no. 5, pp. 910-927, 2000.

[7] I. Arasaratnam, S. Haykin, and R. J. Elliott, "Discrete-time nonlinear filtering algorithms using Gauss-Hermite quadrature," *Proceedings of the IEEE*, vol. 95, no. 5, pp. 953-977, 2007.

[8] I. Arasaratnam, S. Haykin, "Cubature Kalman filters," *IEEE Transactions on Automatic Cotrol*, vol. 54, no. 6, pp. 1254-1269, 2009.

[9] I. Arasaratnam, S. Haykin, and Thomas R. Hurd "Cubature Kalman filtering for continous-discrete systems: theory and simulations," *IEEE Transactions on Signal Processing*, vol. 58, no. 10, pp. 4977-4993, 2010.

[10] M. S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutoriial on particle filters for online nonlinear/non-Gaussian bayesian tracking," *IEEE Transactions on Signal Processing*, vol. 50, no. 3, pp. 174-188, 2002.

[11] X. R. Li, Z. L. Zhao, "Evaluation of estimation algorithms—part I: incomprehensive measures of performance," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 42, no. 4, pp. 1340-1358, 2006.

[12] Hanlin Yin, X. R. Li, and Jian Lan, "Iterative mid-Range with application to estimation performance evaluation," *IEEE Signal Processing Letters*, vol. 22, no. 11, pp. 2044-2048, 2015.

[13] X. R. Li, Z. L. Zhao, and X. Bai Li, "Evaluation of estimation algorithms: credibility tests," *IEEE Transactions on Systems Man and Cybernetics— Part A Systems and Humans*, vol. 42, no. 1, pp. 147-163, 2012.

[14] X. R. Li, Z. L. Zhao, "Measuring estimator's credibility: noncredibility index," *In Proceedings of the 9th International Conference on Information Fusion*, Florence, Italy, pp. 1-7, July 2006.

[15] H. Zhou, H. Yang, "Application of weighted principal component analysis in comprehensive evaluation for power quality," *In Proceedings of the 2011 International Conference on Power Engineering and Automation*, Wuhan, China, pp. 369-372, Sept 2011.

[16] L. F. Shi, "Entropy based fuzzy comprehensive evaluation of university teachers," *In Proceedings of the 2010 International Conference on Pervasive Computing Signal Processing and Applications*, Harbin, China, pp. 475-478, Sept 2010.

[17] Zhi-Ying Lv, Xue-Zhang Liang, "A fuzzy multiple attribute decision making method based on possibility degree," *Fuzzy Systems and Knowledge Discovery (FSKD), 2015 12th International Conference on*, Zhangjiajie, China, pp. 450-454, Aug 2015.

[18] H. L. Yin, J. Lan, and X. R. Li, "Measures for ranking estimation performance based on single or multiple performance metrics," *In Proceedings of the 16th International Conference on Information Fusion*, Istanbul, Turkey, pp. 453-460, July 2013.

[19] X. R. Li, Z. L. Zhao, and Z. S. Duan, "Error spectrum and desirability level for estimation performance evaluation," *In Proceedings of 2007 workshop on Estimation ,Tracking and Fusion : A Trbute to Fred Daum*, Monterey, CA, USA, May 2007.

[20] Yanhui Mao, Chongzhao Han, and Zhansheng Duan, "Dynamic error spectrum for estimation performance evaluation: a case study on interacting multiple model algorithm," *IET Signal Processing*, vol. 8, no. 2, pp. 202-210, 2014.

[21] Arrow, Kenneth J, *Social choice and individual values (2nd ed.)*, New Haven: Yale University Press, 1963.

[22] E. J. G. Pitman, "The closest estimates of statistical parameters," *Mathematical Proceedings of the Cabridge Philosophical Society*, vol. 33, no. 33, pp. 212-222, 1937.

[23] M. Ghosh, S. P. Kumar, "Bayesian Pitman closeness," *Communications in Statistics— Theory and Methods*, vol. 20, no. 11, pp. 3659-3678, 1991.

[24] J. P. Keating, R. L. Mason, and P. K. Sen, *Pitman's measure of closeness: a comparison of statistical estimators*, Philadelphia: Society for Industrial and Applied Mathematics, 1993.

[25] H. L. Yin, J. Lan, and X. R. Li, "Ranking estimation performance by estimator randomization and attribute support," *In Proceedings of the 17th International Conference on Information Fusion*, Salamanca, Spain, pp. 1-8, July 2014.

[26] C. Meyer, *Matrix analysis and applied linear algebra*, Philadelphia: Society for Industrial and Applied Mathematics, 2000.

[27] Peter C. Fishburn, "Condorcet social choice functions," *SIAM Journal on Applied Mathematics*, Vol. 33, No. 3, pp. 469-489, 1977.

[28] David Lippman, *Math in society (2.4 ed.)*, CreateSpace Independent Publishing Platform, 2012.