Inverse Covariance Intersection: New Insights and Properties

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***Abstract*—Decentralized data fusion is a challenging task. Either it is too difficult to maintain and track the information required to perform fusion optimally, or too much information is discarded to obtain informative fusion results. A well-known solution is Covariance Intersection, which may provide too conservative fusion results. A less conservative alternative is discussed in this paper, and generalizations are proposed in order to apply it to a wide class of fusion problems. The Inverse Covariance Intersection algorithm is about finding the maximum possible common information shared by the estimates to be fused. A bound on the possibly shared common information is derived and removed from the fusion result in order to guarantee consistency. It is shown that the conditions required for consistency can be significantly relaxed, and also other causes of correlations, such as common process noise, can be treated.**

***Index Terms*—Decentralized data fusion, covariance intersection, ellipsoidal intersection, track-to-track fusion**

Notation

An underlined variable $\underline{x} \in \mathbb{R}^n$ denotes a real-valued vector. Lowercase boldface letters $\underline{\mathbf{x}}$ are used for random quantities. Matrices are written in uppercase boldface letters $\mathbf{C} \in \mathbb{R}^{n\times n}$, and $\mathbf{C}^{-1}$ and $\mathbf{C}^{\mathrm{T}}$ are its inverse and transpose, respectively. $\mathbf{C} \geq \mathbf{C}'$ implies that the difference $\mathbf{C} - \mathbf{C}'$ is positive semi-definite. The notation $(\hat{\underline{\mathbf{x}}}, \mathbf{C})$ is used for an estimate $\hat{\underline{\mathbf{x}}}$ of $\underline{\mathbf{x}}$, which has the error covariance matrix $\mathbf{C} = \mathrm{E}[\tilde{\underline{\mathbf{x}}}\,\tilde{\underline{\mathbf{x}}}^{\mathrm{T}}]$ with the estimation error $\tilde{\underline{\mathbf{x}}} = \underline{\mathbf{x}} - \hat{\underline{\mathbf{x}}}$. $\mathcal{E}(\hat{\underline{c}}, \mathbf{X}) = \{\underline{x} \in \mathbb{R}^n \mid (\underline{x} - \hat{\underline{c}})^{\mathrm{T}}\mathbf{X}^{-1}(\underline{x} - \hat{\underline{c}})\}$ denotes an ellipsoid with center $\hat{\underline{c}}$ and shape matrix $\mathbf{X}$.

I. Introduction

Decentralized data fusion deals with finding a trade-off between consistent and informative estimates [1]. Optimal fusion that minimizes the uncertainty about the result is difficult to implement as it requires particular information about the estimates to be fused [2]. As such, the well-known Bar-Shalom/Campo fusion formulae [3] depend on the cross-covariance matrix of the estimation errors. Their generalization to the simultaneous fusion of multiple estimates [4], [5] relies on the joint error covariance matrix of the corresponding estimation errors. Again, the cross-covariance matrix between each pair of estimates is required. Storing and keeping track of cross-covariance information is cumbersome and sometimes impossible if sensor nodes are supposed to operate independently. The explicit treatment of cross-covariance information can be circumvented if a specific decomposition of the Kalman filter formulae is exploited [6], [7], or frequent communication takes place [8]. Although these methods can be even optimal in the sense of a single, centralized Kalman filter, rather restrictive prerequisites have to be met, which may hinder their application in complex network architectures. In particular, fully decentralized networks are therefore in need of fusion techniques that do not rely on cross-covariance matrices or specific requirements on the communication policy. In this paper, a novel technique to fuse estimates with an unknown cross-covariance structure is discussed and further developed.

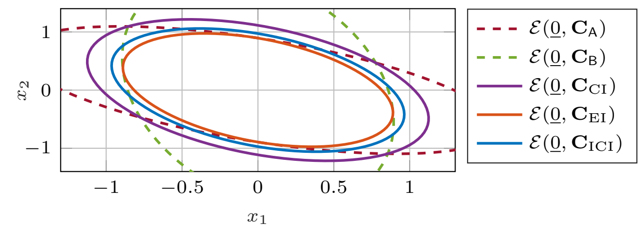


Fig. 1: Results of different fusion methods depicted as ellipsoids centered at origin. The ellipsoid related to CI is a bound on the intersection $\mathcal{E}(\underline{0}, \mathbf{C}_{\mathsf{A}}) \cap \mathcal{E}(\underline{0}, \mathbf{C}_{\mathsf{B}})$. The ellipsoid related to EI is the largest ellipsoid inside the intersection. The ellipsoid for ICI lies in between those related to CI and EI.

A. State-of-the-art Intersection

Instead of striving for an optimal fusion result, a different fusion strategy is to conservatively bound missing or discarded cross-covariance information; as a consequence, this information does not need to be maintained or reconstructed. In this respect, *Covariance Intersection* (CI) [9], [10] is probably the most well-known example, which provides the fusion result

$$\hat{\underline{\mathbf{x}}}_{\mathrm{CI}} = \mathbf{K}_{\mathrm{CI}}\,\hat{\underline{\mathbf{x}}}_{\mathsf{A}} + \mathbf{L}_{\mathrm{CI}}\,\hat{\underline{\mathbf{x}}}_{\mathsf{B}} \tag{1a}$$

with covariance matrix

$$\mathbf{C}_{\mathrm{CI}} = \left((1-\omega)\mathbf{C}_{\mathsf{A}}^{-1} + \omega\mathbf{C}_{\mathsf{B}}\right)^{-1} \tag{1b}$$

and $\omega \in [0,1]$ for the estimates $(\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $(\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$. The gains in (1a) are given by $\mathbf{K}_{\mathrm{CI}} = (1-\omega)\mathbf{C}_{\mathrm{CI}}\mathbf{C}_{\mathsf{A}}^{-1}$ and $\mathbf{L}_{\mathrm{CI}} = \omega\mathbf{C}_{\mathrm{CI}}\mathbf{C}_{\mathsf{B}}^{-1}$. Its name originates from the observation shown in Fig. 1, where the ellipsoid $\mathcal{E}(\underline{0}, \mathbf{C}_{\mathrm{CI}})$ related to the

CI covariance matrix represents a bound on the intersection $\mathcal{E}(\underline{0}, \mathbf{C}_\mathsf{A}) \cap \mathcal{E}(\underline{0}, \mathbf{C}_\mathsf{B})$. CI provides *consistent* fusion results, i.e., $\mathbf{C}_{\mathrm{CI}} \geq \mathrm{E}[\underline{\tilde{\mathbf{x}}}_{\mathrm{CI}} \underline{\tilde{\mathbf{x}}}_{\mathrm{CI}}^{\mathrm{T}}]$ with $\underline{\tilde{\mathbf{x}}}_{\mathrm{CI}} = \underline{\mathbf{x}} - \underline{\hat{\mathbf{x}}}_{\mathrm{CI}}$, given that the estimates $(\underline{\hat{\mathbf{x}}}_\mathsf{A}, \mathbf{C}_\mathsf{A})$ and $(\underline{\hat{\mathbf{x}}}_\mathsf{B}, \mathbf{C}_\mathsf{B})$ to be fused are consistent. Consistency implies that the reported covariance matrix is an upper bound of the actual error covariance matrix.

Since CI has emerged as a universal tool to fuse estimates, much effort to improve and extend CI has been expended in the last two decades. Improvements include the efficient computation of the fusion result, which typically requires a numerical optimization of the weighting parameter ω. These are suboptimal but fast solutions [11], [12], specific optimization criteria [13], [14], or closed-form solutions [15] for special cases. Robustness and stability have been widely studied for CI against the background of communication constraints [16], data-driven policies [17], diffusion strategies [18], and heterogeneous state representations [19]. The effect of multiple processing steps on the fusion quality has been discussed in [20], as well as the treatment and fusion of multiple estimates have been widely studied [21], [22]. The possibility to express CI in terms of probability densities, i.e., exponential mixture densities [23], [24], has also been widely exploited. In doing so, CI fusion can be applied to Gaussian mixtures [25] or PHD filters [26]. For the fusion of densities, also a suitable weighting parameter has to be computed [27], [28]. Applying CI to density functions, however, still requires research toward a proper understanding of general conservativeness as it is conducted in [29].

B. On Optimality of Covariance Intersection

Conservativeness for the linear fusion problem in (1) is well understood, and it has even been proven in [30], [31] that CI can be deemed to be the optimal fusion rule if the cross-covariance matrices are unknown and cannot be exploited. Optimality means that (1a) is the fusion result with the smallest bound (1b) on its error covariance matrix. More precisely, the error covariance matrix for (1a) is given by

$$\begin{aligned} \mathrm{E}[\underline{\tilde{\mathbf{x}}}_{\mathrm{CI}} \underline{\tilde{\mathbf{x}}}_{\mathrm{CI}}^{\mathrm{T}}] = \mathbf{K}_{\mathrm{CI}} \mathbf{C}_\mathsf{A} \mathbf{K}_{\mathrm{CI}}^{\mathrm{T}} + \mathbf{K}_{\mathrm{CI}} \mathbf{C}_\mathsf{AB} \mathbf{L}_{\mathrm{CI}}^{\mathrm{T}} \\ + \mathbf{L}_{\mathrm{CI}} \mathbf{C}_\mathsf{BA} \mathbf{K}_{\mathrm{CI}}^{\mathrm{T}} + \mathbf{L}_{\mathrm{CI}} \mathbf{C}_\mathsf{B} \mathbf{L}_{\mathrm{CI}}^{\mathrm{T}} \end{aligned} \tag{2}$$

and depends on the actual but unknown cross-covariance terms $\mathbf{C}_\mathsf{AB} = \mathbf{C}_\mathsf{BA}^{\mathrm{T}} = \mathrm{E}[\underline{\tilde{\mathbf{x}}}_\mathsf{A} \underline{\tilde{\mathbf{x}}}_\mathsf{B}^{\mathrm{T}}]$. The matrix $\mathbf{C}_{\mathrm{CI}}$ is the smallest bound[1] on $\mathrm{E}[\underline{\tilde{\mathbf{x}}}_{\mathrm{CI}} \underline{\tilde{\mathbf{x}}}_{\mathrm{CI}}^{\mathrm{T}}] \leq \mathbf{C}_{\mathrm{CI}}$ that holds for each possible $\mathbf{C}_\mathsf{AB}$. This result is a strong argument in favor of CI and yet, research is also directed toward alternative fusion methods.

C. Why Yet Another Intersection?

Aside from a plethora of extensive studies on CI, alternative approaches have been proposed that strive for fusion results with a smaller error covariance matrix. An important candidate is *Ellipsoidal Intersection* (EI) [32], which computes the result

$$\underline{\hat{\mathbf{x}}}_{\mathrm{EI}} = \mathbf{C}_{\mathrm{EI}} \big(\mathbf{C}_\mathsf{A}^{-1} \underline{\hat{\mathbf{x}}}_\mathsf{A} + \mathbf{C}_\mathsf{B}^{-1} \underline{\hat{\mathbf{x}}}_\mathsf{B} - \mathbf{\Gamma}_{\mathrm{EI}}^{-1} \underline{\hat{\gamma}}_{\mathrm{EI}} \big) \tag{3a}$$

with covariance matrix

$$\mathbf{C}_{\mathrm{EI}}^{-1} = \mathbf{C}_\mathsf{A}^{-1} + \mathbf{C}_\mathsf{B}^{-1} - \mathbf{\Gamma}_{\mathrm{EI}}^{-1} \ . \tag{3b}$$

[1]This is valid for each $\omega \in [0, 1]$, see [31].

The parameters $\underline{\hat{\gamma}}_{\mathrm{EI}}$ and $\mathbf{\Gamma}_{\mathrm{EI}}$ are designed to account for the maximum possible common information shared by the estimates to be fused. EI proves its usefulness in several theoretical [33] and practical [34], [35] case studies. In [36], it has been shown that EI corresponds to the largest ellipsoid within the intersection, as it can be seen in Fig. 1. As the same figure reveals, a motivation behind EI is to provide a smaller covariance matrix than CI does, but consistency then poses an issue that has to be addressed.

II. Inverse Covariance Intersection

The optimality aspects of CI suggest that alternative fusion rules can only provide consistent results under restrictive conditions. In order to attain less conservative but still consistent estimates, the matrix $\mathbf{C}_\mathsf{AB}$ in (2) must fulfill specific requirements. A typical example for a specific correlation structure is common information that is shared by the nodes in a sensor network [1], [37]. For this case—as shown in [38]—CI is too conservative to optimally treat correlations caused by unknown common information, on the one hand, and the parameters $\underline{\hat{\gamma}}_{\mathrm{EI}}$ and $\mathbf{\Gamma}_{\mathrm{EI}}$ in (3), which have been originally used for EI, are not sufficient to guarantee consistency, on the other hand. To bridge the gap between CI and EI, [38] introduces a new set of parameters that provide less conservative but still consistent estimates. The fusion rule in [38] is the optimal way to treat unknown common information; however, the current paper unveils that the same fusion rule can be also applied far beyond unknown common information, e.g., to the problem of common process noise. Before that, this section provides a brief summary of the fusion rule in question.

Given two consistent estimates $(\underline{\hat{\mathbf{x}}}_\mathsf{A}, \mathbf{C}_\mathsf{A})$ and $(\underline{\hat{\mathbf{x}}}_\mathsf{B}, \mathbf{C}_\mathsf{B})$, *Inverse Covariance Intersection* (ICI) provides the fusion result $(\underline{\hat{\mathbf{x}}}_{\mathrm{ICI}}, \mathbf{C}_{\mathrm{ICI}})$ with

$$\underline{\hat{\mathbf{x}}}_{\mathrm{ICI}} = \mathbf{K}_{\mathrm{ICI}} \, \underline{\hat{\mathbf{x}}}_\mathsf{A} + \mathbf{L}_{\mathrm{ICI}} \, \underline{\hat{\mathbf{x}}}_\mathsf{B} \tag{4a}$$

and

$$\mathbf{C}_{\mathrm{ICI}}^{-1} = \mathbf{C}_\mathsf{A}^{-1} + \mathbf{C}_\mathsf{B}^{-1} - \big(\omega \mathbf{C}_\mathsf{A} + (1-\omega) \mathbf{C}_\mathsf{B} \big)^{-1} \tag{4b}$$

for $\omega \in [0, 1]$. The gains in (4a) are given by

$$\mathbf{K}_{\mathrm{ICI}} = \mathbf{C}_{\mathrm{ICI}} \cdot \big(\mathbf{C}_\mathsf{A}^{-1} - \omega (\omega \mathbf{C}_\mathsf{A} + (1-\omega) \mathbf{C}_\mathsf{B})^{-1} \big) \ , \tag{5a}$$

$$\mathbf{L}_{\mathrm{ICI}} = \mathbf{C}_{\mathrm{ICI}} \cdot \big(\mathbf{C}_\mathsf{B}^{-1} - (1-\omega) (\omega \mathbf{C}_\mathsf{A} + (1-\omega) \mathbf{C}_\mathsf{B})^{-1} \big) . \tag{5b}$$

A simple MATLAB implementation can be downloaded from https://github.com/KIT-ISAS/ICI.

ICI is a novel approach to treat unknown correlations between the estimates to be fused. Since ICI is tailored to a specific correlation structure, less conservative bounds on the fused error covariance matrix are provided than CI can compute. More precisely, it has been shown in [38] that (4b) is smaller than (1b) for each $\omega \in [0, 1]$, i.e., $\mathbf{C}_{\mathrm{ICI}}(\omega) \leq \mathbf{C}_{\mathrm{CI}}(\omega)$.

In the following subsection, we recall the conditions that have been utilized in [38] to prove consistency of the ICI fusion rule. Under these conditions, the covariance matrix (4b) is a conservative bound on the actual error covariance matrix, i.e.,

$$\mathrm{E}\left[\underline{\tilde{\mathbf{x}}}_{\mathrm{ICI}}\underline{\tilde{\mathbf{x}}}_{\mathrm{ICI}}^{\mathrm{T}}\right] = \mathrm{E}\left[(\underline{\mathbf{x}}-\underline{\hat{\mathbf{x}}}_{\mathrm{ICI}})(\underline{\mathbf{x}}-\underline{\hat{\mathbf{x}}}_{\mathrm{ICI}})^{\mathrm{T}}\right] \leq \mathbf{C}_{\mathrm{ICI}} \tag{6}$$

for each $\omega \in [0,1]$. Sec. III will reveal that consistency also holds under far weaker conditions and enables us to apply ICI to more general fusion problems.

A. Considered Correlation Structure

For the derivation of ICI, a parameterization similar to EI has been exploited. In the spirit of [38], we assume that the considered estimates $(\underline{\hat{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $(\underline{\hat{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$ to be fused are consistent and share the common estimate $(\underline{\hat{\gamma}}, \mathbf{\Gamma})$. In particular, both estimates can be represented by the mutually uncorrelated partial estimates $(\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{A}}, \mathbf{\Lambda}_{\mathsf{A}})$, $(\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{B}}, \mathbf{\Lambda}_{\mathsf{B}})$, and $(\underline{\hat{\gamma}}, \mathbf{\Gamma})$ according to

$$\underline{\hat{\mathbf{x}}}_{\mathsf{A}} = \mathbf{C}_{\mathsf{A}}\left(\mathbf{\Lambda}_{\mathsf{A}}^{-1}\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{A}} + \mathbf{\Gamma}^{-1}\underline{\hat{\gamma}}\right) , \tag{7a}$$

$$\mathbf{C}_{\mathsf{A}} = \left(\mathbf{\Lambda}_{\mathsf{A}}^{-1} + \mathbf{\Gamma}^{-1}\right)^{-1} , \tag{7b}$$

and

$$\underline{\hat{\mathbf{x}}}_{\mathsf{B}} = \mathbf{C}_{\mathsf{B}}\left(\mathbf{\Lambda}_{\mathsf{B}}^{-1}\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{B}} + \mathbf{\Gamma}^{-1}\underline{\hat{\gamma}}\right) , \tag{8a}$$

$$\mathbf{C}_{\mathsf{B}} = \left(\mathbf{\Lambda}_{\mathsf{B}}^{-1} + \mathbf{\Gamma}^{-1}\right)^{-1} . \tag{8b}$$

The representations (7) and (8) correspond to the information form [39], i.e., inverse covariance formulation, where $(\underline{\hat{\gamma}}, \mathbf{\Gamma})$ has been fused with both $(\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{A}}, \mathbf{\Lambda}_{\mathsf{A}})$ and $(\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{B}}, \mathbf{\Lambda}_{\mathsf{B}})$. This means that (7) and (8) can each be regarded as the optimal fusion result of uncorrelated estimates:

- In (7), $(\underline{\hat{\gamma}}, \mathbf{\Gamma})$ has been fused with $(\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{A}}, \mathbf{\Lambda}_{\mathsf{A}})$.
- In (8), $(\underline{\hat{\gamma}}, \mathbf{\Gamma})$ has been fused with $(\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{B}}, \mathbf{\Lambda}_{\mathsf{B}})$.

The decompositions (7) and (8) carry over to the corresponding estimation errors, which yield

$$\begin{aligned}\underline{\tilde{\mathbf{x}}}_{\mathsf{A}} = \underline{\hat{\mathbf{x}}}_{\mathsf{A}} - \underline{\mathbf{x}} &= \mathbf{C}_{\mathsf{A}}\left(\mathbf{\Lambda}_{\mathsf{A}}^{-1}\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{A}} + \mathbf{\Gamma}^{-1}\underline{\hat{\gamma}}\right) - \underline{\mathbf{x}} \\ &= \mathbf{C}_{\mathsf{A}}\left(\mathbf{\Lambda}_{\mathsf{A}}^{-1}(\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{A}} - \underline{\mathbf{x}}) + \mathbf{\Gamma}^{-1}(\underline{\hat{\gamma}} - \underline{\mathbf{x}})\right) \\ &= \mathbf{C}_{\mathsf{A}}\left(\mathbf{\Lambda}_{\mathsf{A}}^{-1}\underline{\tilde{\boldsymbol{\lambda}}}_{\mathsf{A}} + \mathbf{\Gamma}^{-1}\underline{\tilde{\gamma}}\right)\end{aligned}$$

and

$$\underline{\tilde{\mathbf{x}}}_{\mathsf{B}} = \mathbf{C}_{\mathsf{B}}\left(\mathbf{\Lambda}_{\mathsf{B}}^{-1}\underline{\tilde{\boldsymbol{\lambda}}}_{\mathsf{B}} + \mathbf{\Gamma}^{-1}\underline{\tilde{\gamma}}\right) .$$

Since the partial estimates have been assumed to be uncorrelated, i.e., $\mathrm{E}[\underline{\tilde{\boldsymbol{\lambda}}}_{\mathsf{A}}\underline{\tilde{\boldsymbol{\lambda}}}_{\mathsf{B}}^{\mathrm{T}}] = \mathrm{E}[\underline{\tilde{\boldsymbol{\lambda}}}_{\mathsf{A}}\underline{\tilde{\gamma}}^{\mathrm{T}}] = \mathrm{E}[\underline{\tilde{\boldsymbol{\lambda}}}_{\mathsf{B}}\underline{\tilde{\gamma}}^{\mathrm{T}}] = \mathbf{0}$, the error cross-covariance matrix $\mathbf{C}_{\mathsf{AB}}$ of $\underline{\tilde{\mathbf{x}}}_{\mathsf{A}}$ and $\underline{\tilde{\mathbf{x}}}_{\mathsf{B}}$ becomes

$$\mathbf{C}_{\mathsf{AB}} = \mathbf{C}_{\mathsf{BA}}^{\mathrm{T}} = \mathrm{E}\left[\underline{\tilde{\mathbf{x}}}_{\mathsf{A}}\underline{\tilde{\mathbf{x}}}_{\mathsf{B}}^{\mathrm{T}}\right] = \mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}^{-1}\mathbf{C}_{\mathsf{B}} . \tag{10}$$

If we assume for a moment that $(\underline{\hat{\gamma}}, \mathbf{\Gamma})$ is known to the fusion center, we are in the position to compute the optimal fusion result $(\underline{\hat{\mathbf{x}}}_{\mathrm{opt}}, \mathbf{C}_{\mathrm{opt}})$ given by

$$\begin{aligned}\underline{\hat{\mathbf{x}}}_{\mathrm{opt}} &= \mathbf{C}_{\mathrm{opt}}\left(\mathbf{\Lambda}_{\mathsf{A}}^{-1}\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{A}} + \mathbf{\Lambda}_{\mathsf{B}}^{-1}\underline{\hat{\boldsymbol{\lambda}}}_{\mathsf{B}} + \mathbf{\Gamma}^{-1}\underline{\hat{\gamma}}\right) \\ &= \mathbf{C}_{\mathrm{opt}}\left(\mathbf{C}_{\mathsf{A}}^{-1}\underline{\hat{\mathbf{x}}}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}}^{-1}\underline{\hat{\mathbf{x}}}_{\mathsf{B}} - \mathbf{\Gamma}^{-1}\underline{\hat{\gamma}}\right)\end{aligned} \tag{11a}$$

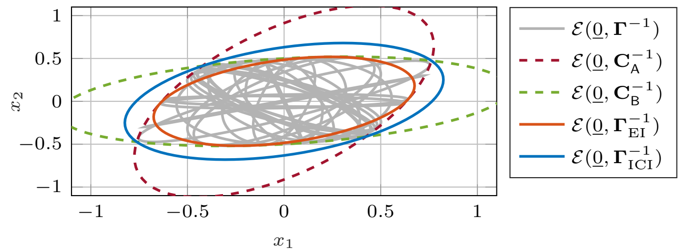


Fig. 2: Bound on the ellipsoids that represent possible common estimates. 70 possible inverse covariance matrices $\mathbf{\Gamma}^{-1}$ are shown. A bound on all $\mathcal{E}(\underline{0}, \mathbf{\Gamma}^{-1})$ has to circumscribe the intersection $\mathcal{E}(\underline{0}, \mathbf{C}_{\mathsf{A}}^{-1}) \cap \mathcal{E}(\underline{0}, \mathbf{C}_{\mathsf{B}}^{-1})$. This property holds for $\mathbf{\Gamma}_{\mathrm{ICI}}^{-1}$, but $\mathbf{\Gamma}_{\mathrm{EI}}^{-1}$ in (3b) has been chosen too small.

and

$$\begin{aligned}\mathbf{C}_{\mathrm{opt}}^{-1} &= \mathbf{\Lambda}_{\mathsf{A}}^{-1} + \mathbf{\Lambda}_{\mathsf{B}}^{-1} + \mathbf{\Gamma}^{-1} \\ &= \mathbf{C}_{\mathsf{A}}^{-1} + \mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}^{-1} .\end{aligned} \tag{11b}$$

This corresponds to the fusion of three uncorrelated estimates in the information form. The latter part in each equation reveals that common information can simply be removed from the naïve[2] fusion result by subtraction—a technique that is exploited by the channel filter [37]. In contrast to the channel filter, ICI is concerned with the case that common information is *unknown*.

B. Intuition Behind Inverse Covariance Intersection

In order to treat unknown common information $(\underline{\hat{\gamma}}, \mathbf{\Gamma})$, the ICI fusion rule (4) rests upon the idea to subtract a bound on the *maximum possible* common information in (11). As it can be seen in (4b), the used upper bound on the inverse covariance matrix $\mathbf{\Gamma}^{-1}$ is

$$\mathbf{\Gamma}^{-1} \leq \left(\omega\mathbf{C}_{\mathsf{A}} + (1-\omega)\mathbf{C}_{\mathsf{B}}\right)^{-1} =: \mathbf{\Gamma}_{\mathrm{ICI}}^{-1} , \tag{12}$$

for each $\omega \in [0,1]$. This relationship holds for each set of possible parameters in (7) and (8) as it is illustrated in Fig. 2, which also reveals that the right-hand side in (12) corresponds to an outer ellipsoidal approximation of the intersection

$$\mathcal{E}(\underline{0}, \mathbf{C}_{\mathsf{A}}^{-1}) \cap \mathcal{E}(\underline{0}, \mathbf{C}_{\mathsf{B}}^{-1}) \subseteq \mathcal{E}(\underline{0}, \mathbf{\Gamma}_{\mathrm{ICI}}^{-1}) ,$$

which gives ICI its name. As it has been proven in [38], the bound is even tight and hence, ICI constitutes the optimal way to treat unknown common information. However, this result is only half of the story as Sec. III will reveal.

C. Illustrative Example

For the purpose of comparing the different intersection methods, we revisit the example in [32]. The estimates to be fused have the parameters

$$\begin{aligned}(\underline{\hat{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}}) &= \left(\begin{bmatrix}0.5\\1\end{bmatrix}, \begin{bmatrix}2.5 & -1\\-1 & 1.2\end{bmatrix}\right) , \\ (\underline{\hat{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}}) &= \left(\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}0.8 & -0.5\\-0.5 & 4\end{bmatrix}\right) ,\end{aligned}$$

and Fig. 3 depicts the fusion results provided by the EI, CI, and ICI algorithms. The covariance matrices reported by the fusion

[2] A naïve fusion ignores common information and possible correlations.

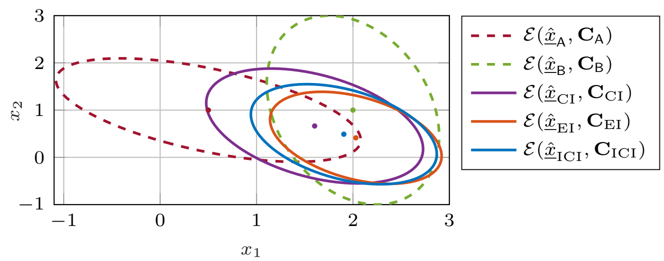


Fig. 3: Fusion results of the estimates $(\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $(\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$. The corresponding ellipsoids centered at the origin are shown in Fig. 1, and the ellipsoids for possibly shared common information are discussed in Fig. 2.

methods are compared in Fig. 1, where the corresponding covariance ellipsoids are centered at the origin.

The ICI approach attains an estimate that is close to the EI result while being still less conservative than CI. For EI, an estimate $(\hat{\underline{\gamma}}_{\mathrm{EI}}, \mathbf{\Gamma}_{\mathrm{EI}})$ is computed that is subtracted in (11). EI provides the smallest error ellipse, but consistency is not guaranteed for the case of unknown common information, which is related to the observation in Fig. 2, where $\mathbf{\Gamma}_{\mathrm{EI}}^{-1}$ is too small to account for all possible common estimates. By contrast, ICI provides a tight bound for all possible common estimates [38]. While ICI is based on a bound on the intersection for the inverse covariance matrices in Fig. 2, CI computes a bound directly on the intersection of the covariance ellipsoids as it can be see in Fig. 1. Evidently, CI is too conservative for the considered correlation structure (10).

ICI is tailored to the specific correlation structure (10). While this allows us to compute less conservative fusion results, it is an intriguing question whether ICI is still consistent when correlations differ from (10). The following considerations offer an answer to this question.

III. New Properties and Insights

For the purpose of generalizing ICI, we do not have to alter the fusion rule (4) itself but consider a modified problem structure different to (7) and (8). In doing so, ICI becomes applicable to more general fusion problems, and it can even be employed to treat the problem of common process noise, which is discussed in Sec. IV.

The fusion result (4) provided by ICI is consistent if condition (6) is fulfilled. This has already been proven for estimates that share a common estimate and are given in the form (7) and (8). In this section, two key generalizations of the considered problem structure are proposed:

(G1) The estimates to be fused may share *correlated* information, not only *common* information.

(G2) The shared information has to be included only in *one* estimate, not necessarily in *both*.

With the aid of these generalizations, we can leverage ICI to guarantee consistency for a far wider class of fusion problems.

In place of the parameterizations (7) and (8), we consider the estimates

$$\hat{\underline{\mathbf{x}}}_{\mathsf{A}} = \mathbf{C}_{\mathsf{A}}\left(\mathbf{\Lambda}_{\mathsf{A}}^{-1}\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}} + \mathbf{\Gamma}_{\mathsf{A}}^{-1}\hat{\underline{\gamma}}_{\mathsf{A}}\right) , \tag{13a}$$
$$\mathbf{C}_{\mathsf{A}} = \left(\mathbf{\Lambda}_{\mathsf{A}}^{-1} + \mathbf{\Gamma}_{\mathsf{A}}^{-1}\right)^{-1} , \tag{13b}$$

and

$$\hat{\underline{\mathbf{x}}}_{\mathsf{B}} = \mathbf{C}_{\mathsf{B}}\left(\mathbf{\Lambda}_{\mathsf{B}}^{-1}\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}} + \mathbf{\Gamma}_{\mathsf{B}}^{-1}\hat{\underline{\gamma}}_{\mathsf{B}}\right) , \tag{14a}$$
$$\mathbf{C}_{\mathsf{B}} = \left(\mathbf{\Lambda}_{\mathsf{B}}^{-1} + \mathbf{\Gamma}_{\mathsf{B}}^{-1}\right)^{-1} . \tag{14b}$$

Again, it is assumed that the partial estimates $(\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}}, \mathbf{\Lambda}_{\mathsf{A}})$, $(\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}}, \mathbf{\Lambda}_{\mathsf{B}})$, $(\hat{\underline{\gamma}}_{\mathsf{A}}, \mathbf{\Gamma}_{\mathsf{A}})$ and $(\hat{\underline{\gamma}}_{\mathsf{B}}, \mathbf{\Gamma}_{\mathsf{B}})$ are mutually uncorrelated—except for the latter pair, which has the cross-covariance matrix

$$\mathbf{\Gamma}_{\mathsf{AB}} = \mathbf{\Gamma}_{\mathsf{BA}}^{\mathrm{T}} = \mathrm{E}\left[\tilde{\underline{\gamma}}_{\mathsf{A}}\tilde{\underline{\gamma}}_{\mathsf{B}}^{\mathrm{T}}\right] .$$

Compared to (7) and (8), the estimates (13) and (14) are not required to share common information but correlated information. As a consequence, the cross-covariance matrix of (13) and (14) becomes

$$\begin{aligned}\mathbf{C}_{\mathsf{AB}} = \mathbf{C}_{\mathsf{BA}}^{\mathrm{T}} &= \mathrm{E}\left[\tilde{\underline{\mathbf{x}}}_{\mathsf{A}}\tilde{\underline{\mathbf{x}}}_{\mathsf{B}}^{\mathrm{T}}\right] \\ &= \mathrm{E}\left[\mathbf{C}_{\mathsf{A}}\left(\mathbf{\Lambda}_{\mathsf{A}}^{-1}\tilde{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}} + \mathbf{\Gamma}_{\mathsf{A}}^{-1}\tilde{\underline{\gamma}}_{\mathsf{A}}\right)\left(\mathbf{\Lambda}_{\mathsf{B}}^{-1}\tilde{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}} + \mathbf{\Gamma}_{\mathsf{B}}^{-1}\tilde{\underline{\gamma}}_{\mathsf{B}}\right)^{\mathrm{T}}\mathbf{C}_{\mathsf{B}}\right] \\ &= \mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{\Gamma}_{\mathsf{AB}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{B}} ,\end{aligned} \tag{15}$$

which is different from (10). Based on this generalization, the following considerations will reveal how (G1) and (G2) can be established without impairing consistency.

A. Joint State Space Formulation of Fusion Problem

Before we study the consistency of ICI under the relaxed problem setup (13) and (14), we introduce a joint state space formulation of the ICI fusion rule. We consider the joint estimate $(\hat{\underline{\mathbf{x}}}^{\mathsf{J}}, \mathbf{C}^{\mathsf{J}})$, which comprises both estimates according to

$$\hat{\underline{\mathbf{x}}}^{\mathsf{J}} := \begin{bmatrix}\hat{\underline{\mathbf{x}}}_{\mathsf{A}} \\ \hat{\underline{\mathbf{x}}}_{\mathsf{B}}\end{bmatrix} , \tag{16a}$$
$$\mathbf{C}^{\mathsf{J}} := \begin{bmatrix}\mathbf{C}_{\mathsf{A}} & \mathbf{C}_{\mathsf{AB}} \\ \mathbf{C}_{\mathsf{BA}} & \mathbf{C}_{\mathsf{B}}\end{bmatrix} = \mathrm{E}\left[\begin{bmatrix}\tilde{\underline{\mathbf{x}}}_{\mathsf{A}} \\ \tilde{\underline{\mathbf{x}}}_{\mathsf{B}}\end{bmatrix}\cdot\begin{bmatrix}\tilde{\underline{\mathbf{x}}}_{\mathsf{A}}^{\mathrm{T}} & \tilde{\underline{\mathbf{x}}}_{\mathsf{B}}^{\mathrm{T}}\end{bmatrix}\right] . \tag{16b}$$

As, for instance, stated in [4] or [30], the optimal fusion result can be computed by

$$\hat{\underline{\mathbf{x}}}_{\mathrm{fus}} = \left(\mathbf{H}^{\mathrm{T}}\left(\mathbf{C}^{\mathsf{J}}\right)^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\left(\mathbf{C}^{\mathsf{J}}\right)^{-1}\hat{\underline{\mathbf{x}}}^{\mathsf{J}} , \tag{17a}$$
$$\mathbf{C}_{\mathrm{fus}} = \left(\mathbf{H}^{\mathrm{T}}\left(\mathbf{C}^{\mathsf{J}}\right)^{-1}\mathbf{H}\right)^{-1} \tag{17b}$$

with $\mathbf{H} = \begin{bmatrix}\mathbf{I} & \mathbf{I}\end{bmatrix}^{\mathrm{T}}$. The optimal combination (17) is identical to the Bar-Shalom/Campo fusion rule [3] and can also be interpreted as a weighted least-squares solution [40].

Conservative fusion strategies like CI employ an upper bound on the joint covariance matrix (16b) in order to treat an unknown $\mathbf{C}_{\mathsf{AB}}$. More precisely, the bound [41]

$$\mathbf{C}_{\mathrm{CI}}^{\mathsf{J}} := \begin{bmatrix}\frac{1}{1-\omega}\mathbf{C}_{\mathsf{A}} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\omega}\mathbf{C}_{\mathsf{B}}\end{bmatrix} \geq \mathbf{C}^{\mathsf{J}} \tag{18}$$

holds for each $\omega \in (0,1)$ and replaces $\mathbf{C}^{\mathsf{J}}$ in (17). One can easily check that (17) then results into (1). In general, every upper bound on $\mathbf{C}^{\mathsf{J}}$ provides a consistent fusion results when used in (17).

The derivation of ICI in [38] can also be reformulated in terms of (16) and (17). The corresponding bound for ICI is

$$\mathbf{C}^{\mathsf{J}}_{\mathrm{ICI}} := \begin{bmatrix} \mathbf{C}_{\mathsf{A}} + \frac{1}{\lambda}\mathbf{C}_{\mathsf{A}}\mathbf{C}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\mathsf{B}} + \lambda\mathbf{C}_{\mathsf{B}}\mathbf{C}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{B}} \end{bmatrix} \geq \mathbf{C}^{\mathsf{J}} \tag{19}$$

for each $\lambda > 0$. We do not show that $\mathbf{C}^{\mathsf{J}}_{\mathrm{ICI}}$ is a consistent bound on $\mathbf{C}^{\mathsf{J}}$ in the case of common information as this will be shown for the more general case in the subsequent subsection, but we confirm that $\mathbf{C}^{\mathsf{J}}_{\mathrm{ICI}}$ plugged into (17) is equivalent to (4). For this purpose, we consider (17b), which yields

$$\begin{aligned}
\mathbf{C}_{\mathrm{ICI}}^{-1} &= \mathbf{H}^{\mathrm{T}}\left(\mathbf{C}^{\mathsf{J}}_{\mathrm{ICI}}\right)^{-1}\mathbf{H} \\
&= \left(\mathbf{C}_{\mathsf{A}} + \frac{1}{\lambda}\mathbf{C}_{\mathsf{A}}\mathbf{C}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{A}}\right)^{-1} + \left(\mathbf{C}_{\mathsf{B}} + \lambda\mathbf{C}_{\mathsf{B}}\mathbf{C}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{B}}\right)^{-1} \\
&= \mathbf{C}_{\mathsf{A}}^{-1} - \left(\lambda\mathbf{C}_{\mathsf{B}} + \mathbf{C}_{\mathsf{A}}\right)^{-1} + \mathbf{C}_{\mathsf{B}}^{-1} - \left(\frac{1}{\lambda}\mathbf{C}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}}\right)^{-1} \\
&= \mathbf{C}_{\mathsf{A}}^{-1} + \mathbf{C}_{\mathsf{B}}^{-1} - (1+\lambda)\left(\lambda\mathbf{C}_{\mathsf{B}} + \mathbf{C}_{\mathsf{A}}\right)^{-1} \\
&= \mathbf{C}_{\mathsf{A}}^{-1} + \mathbf{C}_{\mathsf{B}}^{-1} - \left(\frac{\lambda}{1+\lambda}\mathbf{C}_{\mathsf{B}} + \frac{1}{1+\lambda}\mathbf{C}_{\mathsf{A}}\right)^{-1} ,
\end{aligned}$$

by means of the Woodbury formula. By setting $\omega := \frac{1}{1+\lambda}$, we also have $(1-\omega) = \frac{\lambda}{1+\lambda}$ and arrive at (4b). In the same way, it can be shown that using $\mathbf{C}^{\mathsf{J}}_{\mathrm{ICI}}$ in (17a) is identical to (4a).

The considerations in this subsection can be summarized as follows: If the inequality (19) is fulfilled, the ICI estimate (4) is a conservative and consistent fusion result.

B. Relaxed Conditions for Inverse Covariance Intersection

In the following, we study the more general parameterizations (13) and (14). The original proof of consistency in [38] relies on the inequalities

$$\mathbf{\Gamma}^{-1} \leq \mathbf{C}_{\mathsf{A}}^{-1} \quad \text{and} \quad \mathbf{\Gamma}^{-1} \leq \mathbf{C}_{\mathsf{B}}^{-1} , \tag{20}$$

which can easily be seen from (7b) and (8b). For the purpose of applying ICI to the estimates (13) and (14), consistency condition (20) is altered to the inequalities

$$\alpha\mathbf{\Gamma}_{\mathsf{A}}^{-1} \leq \mathbf{C}_{\mathsf{B}}^{-1} \quad \text{and} \quad \frac{1}{\alpha}\mathbf{\Gamma}_{\mathsf{B}}^{-1} \leq \mathbf{C}_{\mathsf{A}}^{-1} , \tag{21}$$

where also a weighting parameter $\alpha > 0$ has been introduced. For $\alpha = 1$ and $\mathbf{\Gamma}_{\mathsf{A}} = \mathbf{\Gamma}_{\mathsf{B}}$, the original formulation given by (7) and (8) can be viewed as a special case. The generalized condition (21) implies that even $\mathbf{\Gamma}_{\mathsf{A}}^{-1} \geq \mathbf{C}_{\mathsf{B}}^{-1}$ $\left(\text{or } \mathbf{\Gamma}_{\mathsf{B}}^{-1} \geq \mathbf{C}_{\mathsf{A}}^{-1}\right)$ may hold as long as there is an $\alpha > 0$ such that (21) is met. In doing so, generalizations (G1) and (G2) can be implemented.

In order to prove consistency of ICI under condition (21), we have to verify that inequality (19) holds irrespective of the actual cross-covariance matrix (15). The considered inequality (19) expresses that the difference

$$\begin{aligned}
&\begin{bmatrix} \mathbf{C}_{\mathsf{A}} + \frac{1}{\lambda}\mathbf{C}_{\mathsf{A}}\mathbf{C}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\mathsf{B}} + \lambda\mathbf{C}_{\mathsf{B}}\mathbf{C}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{B}} \end{bmatrix} - \begin{bmatrix} \mathbf{C}_{\mathsf{A}} & \mathbf{C}_{\mathsf{AB}} \\ \mathbf{C}_{\mathsf{BA}} & \mathbf{C}_{\mathsf{B}} \end{bmatrix} \\
&\quad = \begin{bmatrix} \frac{1}{\lambda}\mathbf{C}_{\mathsf{A}}\mathbf{C}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{A}} & -\mathbf{C}_{\mathsf{AB}} \\ -\mathbf{C}_{\mathsf{BA}} & \lambda\mathbf{C}_{\mathsf{B}}\mathbf{C}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{B}} \end{bmatrix}
\end{aligned} \tag{22}$$

has to be positive definite. The above matrix is bounded from below by

$$\begin{aligned}
&\begin{bmatrix} \frac{1}{\lambda}\mathbf{C}_{\mathsf{A}}\mathbf{C}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{A}} & -\mathbf{C}_{\mathsf{AB}} \\ -\mathbf{C}_{\mathsf{BA}} & \lambda\mathbf{C}_{\mathsf{B}}\mathbf{C}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{B}} \end{bmatrix} \\
&\quad \geq \begin{bmatrix} \frac{\alpha}{\lambda}\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{A}} & -\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{\Gamma}_{\mathsf{AB}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{B}} \\ -\mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{\Gamma}_{\mathsf{BA}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{A}} & \frac{\lambda}{\alpha}\mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{B}} \end{bmatrix} ,
\end{aligned}$$

where (21) has been employed and $\mathbf{C}_{\mathsf{AB}}$ has been replaced by (15). The right-hand side is a positive definite matrix if and only if

$$\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{\alpha}{\lambda}\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{A}} & -\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{\Gamma}_{\mathsf{AB}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{B}} \\ \mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{\Gamma}_{\mathsf{BA}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{A}} & \frac{\lambda}{\alpha}\mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{B}} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} \tag{23}$$

is a positive value for every $\underline{x}_1 \in \mathbb{R}^N$ and $\underline{x}_2 \in \mathbb{R}^N$. With the aid of

$$\underline{y}_1 := \sqrt{\frac{\alpha}{\lambda}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{A}}\underline{x}_1 \text{ and } \underline{y}_2 := -\sqrt{\frac{\lambda}{\alpha}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{B}}\underline{x}_2$$

we obtain

$$\begin{aligned}
&\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{\alpha}{\lambda}\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{A}} & -\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{\Gamma}_{\mathsf{AB}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{B}} \\ -\mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{\Gamma}_{\mathsf{BA}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{A}} & \frac{\lambda}{\alpha}\mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{B}} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} \\
&= \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{\Gamma}_{\mathsf{A}} & \mathbf{\Gamma}_{\mathsf{AB}} \\ \mathbf{\Gamma}_{\mathsf{BA}} & \mathbf{\Gamma}_{\mathsf{B}} \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix}
\end{aligned}$$

Finally, we have

$$\mathbf{\Gamma}^{\mathsf{J}} = \begin{bmatrix} \mathbf{\Gamma}_{\mathsf{A}} & \mathbf{\Gamma}_{\mathsf{AB}} \\ \mathbf{\Gamma}_{\mathsf{BA}} & \mathbf{\Gamma}_{\mathsf{B}} \end{bmatrix} \geq \mathbf{0}$$

since $\mathbf{\Gamma}^{\mathsf{J}}$ is the joint error covariance matrix of the joint estimate $[\hat{\underline{\gamma}}_{\mathsf{A}}^{\mathrm{T}}, \hat{\underline{\gamma}}_{\mathsf{B}}^{\mathrm{T}}]^{\mathrm{T}}$ and thus positive definite. Therefore, (23) is a positive value and thus, the matrix (22) is positive definite, which concludes the proof.

In conclusion, the ICI fusion rule (4) provides consistent results if the estimates (13) and (14) to be fused meet condition (21).

C. Bound on Common Information

The preceding considerations have demonstrated that ICI can be applied to more general estimation problems. In this concluding subsection, we show that the correlated parts of the estimates (13) and (14) still lie in the intersection of $\mathcal{E}(\underline{0}, \mathbf{C}_{\mathsf{A}}^{-1})$ and $\mathcal{E}(\underline{0}, \mathbf{C}_{\mathsf{B}}^{-1})$ after fusion. The fused estimate is given by

$$\begin{aligned}
\hat{\underline{\mathbf{x}}}_{\mathrm{ICI}} &= \mathbf{K}_{\mathrm{ICI}}\,\hat{\underline{\mathbf{x}}}_{\mathsf{A}} + \mathbf{L}_{\mathrm{ICI}}\,\hat{\underline{\mathbf{x}}}_{\mathsf{B}} \\
&= \mathbf{K}_{\mathrm{ICI}}\mathbf{C}_{\mathsf{A}}\mathbf{\Lambda}_{\mathsf{A}}^{-1}\,\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}} + \mathbf{L}_{\mathrm{ICI}}\mathbf{C}_{\mathsf{B}}\mathbf{\Lambda}_{\mathsf{B}}^{-1}\,\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}} \\
&\quad + \mathbf{K}_{\mathrm{ICI}}\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\,\hat{\underline{\gamma}}_{\mathsf{A}} + \mathbf{L}_{\mathrm{ICI}}\mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\,\hat{\underline{\gamma}}_{\mathsf{B}} .
\end{aligned}$$

With the definition of the gains in (5), the estimate can be written as

$$\hat{\underline{\mathbf{x}}}_{\mathrm{ICI}} = \mathbf{C}_{\mathrm{ICI}}\left(\mathbf{\Lambda}_{\mathrm{ICI}}^{-1}\,\hat{\underline{\boldsymbol{\lambda}}}_{\mathrm{ICI}} + \mathbf{\Gamma}_{\mathrm{ICI}}^{-1}\,\hat{\underline{\gamma}}_{\mathrm{ICI}}\right)$$

which resembles the representation (13) or (14). The term $\mathbf{\Lambda}_{\mathrm{ICI}}^{-1}\,\hat{\underline{\boldsymbol{\lambda}}}_{\mathrm{ICI}}$ comprises the independent parts, and the second term $\mathbf{\Gamma}_{\mathrm{ICI}}^{-1}\,\hat{\underline{\gamma}}_{\mathrm{ICI}}$ represents the fused dependent parts. The latter term is studied in the following and a bound is derived.

The fused correlated partial estimates can be rewritten as

$$
\begin{aligned}
\mathbf{\Gamma}_{\mathrm{ICI}}^{-1}\,\underline{\hat{\gamma}}_{\mathrm{ICI}} &= \mathbf{C}_{\mathrm{ICI}}^{-1}\left(\mathbf{K}_{\mathrm{ICI}}\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\,\underline{\hat{\gamma}}_{\mathsf{A}} + \mathbf{L}_{\mathrm{ICI}}\mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\,\underline{\hat{\gamma}}_{\mathsf{B}}\right) \\
&= \bar{\mathbf{K}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\,\underline{\hat{\gamma}}_{\mathsf{A}} + \bar{\mathbf{L}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\,\underline{\hat{\gamma}}_{\mathsf{B}}
\end{aligned} \tag{24}
$$

with the matrices $\bar{\mathbf{K}}$ and $\bar{\mathbf{L}}$ given by

$$
\begin{aligned}
\bar{\mathbf{K}} &= \mathbf{C}_{\mathrm{ICI}}^{-1}\mathbf{K}_{\mathrm{ICI}}\mathbf{C}_{\mathsf{A}} \\
&\overset{(5a)}{=} \left(\mathbf{C}_{\mathsf{A}}^{-1} - \omega(\omega\mathbf{C}_{\mathsf{A}} + (1-\omega)\mathbf{C}_{\mathsf{B}})^{-1}\right)\mathbf{C}_{\mathsf{A}} \\
&= \mathbf{I} - (\omega\mathbf{C}_{\mathsf{A}} + (1-\omega)\mathbf{C}_{\mathsf{B}})^{-1}\,\omega\mathbf{C}_{\mathsf{A}} \\
&= (\omega\mathbf{C}_{\mathsf{A}} + (1-\omega)\mathbf{C}_{\mathsf{B}})^{-1}\,(1-\omega)\mathbf{C}_{\mathsf{B}}\ , \\
\bar{\mathbf{L}} &= (\omega\mathbf{C}_{\mathsf{A}} + (1-\omega)\mathbf{C}_{\mathsf{B}})^{-1}\,\omega\mathbf{C}_{\mathsf{A}}\ .
\end{aligned}
$$

The matrices $\bar{\mathbf{K}}$ and $\bar{\mathbf{L}}$ can be interpreted as fusion gains since $\mathbf{I} = \bar{\mathbf{K}} + \bar{\mathbf{L}}$ holds. The following considerations back up this interpretation. With

$$
\mathbf{\Gamma}_{\mathrm{ICI}}^{-1} = (\omega\mathbf{C}_{\mathsf{A}} + (1-\omega)\mathbf{C}_{\mathsf{B}})^{-1}\ ,
$$

as defined in (12), we can write $\bar{\mathbf{K}} = \mathbf{\Gamma}_{\mathrm{ICI}}^{-1}\,(1-\omega)\mathbf{C}_{\mathsf{B}}$ and $\bar{\mathbf{L}} = \mathbf{\Gamma}_{\mathrm{ICI}}^{-1}\,\omega\mathbf{C}_{\mathsf{A}}$. For the estimation error $\underline{\tilde{x}}_{\mathrm{ICI}}$, we can follow the same calculations as above to obtain the estimation error

$$
\mathbf{\Gamma}_{\mathrm{ICI}}^{-1}\underline{\tilde{\gamma}}_{\mathrm{ICI}} = \bar{\mathbf{K}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\,\underline{\tilde{\gamma}}_{\mathsf{A}} + \bar{\mathbf{L}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\,\underline{\tilde{\gamma}}_{\mathsf{B}}
$$

with respect to (24). The error covariance matrix of $\underline{\tilde{\gamma}}_{\mathrm{ICI}}$ yields

$$
\begin{aligned}
\mathrm{E}[\underline{\tilde{\gamma}}_{\mathrm{ICI}}\underline{\tilde{\gamma}}_{\mathrm{ICI}}^{\mathrm{T}}] = (1-\omega)^2&\mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{B}} \\
&+ (1-\omega)\omega\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{\Gamma}_{\mathsf{AB}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{B}} \\
&+ \omega(1-\omega)\mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{\Gamma}_{\mathsf{BA}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{A}} \\
&+ \omega^2\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{A}}
\end{aligned}
$$

where the cross-covariance terms are given by (15). In order to treat the unknown matrices $\mathbf{\Gamma}_{\mathsf{AB}} = \mathbf{\Gamma}_{\mathsf{BA}}^{\mathrm{T}}$, we exploit the bounding technique (18) to obtain the bound

$$
\begin{aligned}
\mathrm{E}[\underline{\tilde{\gamma}}_{\mathrm{ICI}}\underline{\tilde{\gamma}}_{\mathrm{ICI}}^{\mathrm{T}}] &\leq \frac{(1-\omega)^2}{1-\omega}\mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{B}} + \frac{\omega^2}{\omega}\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{A}} \\
&= (1-\omega)\mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{B}} + \omega\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}_{\mathsf{B}}^{-1}\mathbf{C}_{\mathsf{A}} \\
&\leq (1-\omega)\mathbf{C}_{\mathsf{B}} + \omega\mathbf{C}_{\mathsf{A}}\ .
\end{aligned}
$$

For the latter bound, condition (21) has been employed. To keep it simple, we have assumed that the condition holds for $\alpha = 1$. In information form, we arrive at the bound

$$
\begin{aligned}
\mathrm{E}\left[\mathbf{\Gamma}_{\mathrm{ICI}}^{-1}\underline{\tilde{\gamma}}_{\mathrm{ICI}}\underline{\tilde{\gamma}}_{\mathrm{ICI}}^{\mathrm{T}}\mathbf{\Gamma}_{\mathrm{ICI}}^{-1}\right] &= \mathbf{\Gamma}_{\mathrm{ICI}}^{-1}\,\mathrm{E}[\underline{\tilde{\gamma}}_{\mathrm{ICI}}\underline{\tilde{\gamma}}_{\mathrm{ICI}}^{\mathrm{T}}]\mathbf{\Gamma}_{\mathrm{ICI}}^{-1} \\
\leq \mathbf{\Gamma}_{\mathrm{ICI}}^{-1}(\omega\mathbf{C}_{\mathsf{A}} + (1-\omega)\mathbf{C}_{\mathsf{B}})\mathbf{\Gamma}_{\mathrm{ICI}}^{-1} &= \mathbf{\Gamma}_{\mathrm{ICI}}^{-1}\ .
\end{aligned} \tag{25}
$$

This inequality shows that the actual error covariance matrix in its information form on the left-hand side is bounded by $\mathbf{\Gamma}_{\mathrm{ICI}}^{-1}$. As a result, the inverse error covariance matrix of the fused dependent parts is still related to the intersection of the inverse covariance ellipsoids, as it is stated in (12) and can be seen in Fig. 2.

The bounding inverse covariance matrix $\mathbf{\Gamma}_{\mathrm{ICI}}^{-1}$ for the fused dependent parts is an astonishing result as—at the same time—it represents the maximum common information removed from the fusion result in (4b) and the maximum common information that is still contained in the fusion result as it can be seen from (25).

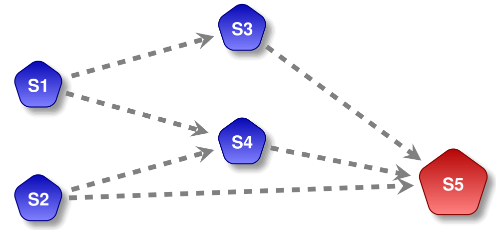


Fig. 4: Communication in sensor network.

IV. A First Application Scenario

In its original formulation, ICI has been designed to fuse estimates that share a common estimate $(\underline{\hat{\gamma}}, \mathbf{\Gamma})$ and as such, the treatment of common process noise [3] has not been considered. The derivations in [38] may even suggest that consistency cannot be guaranteed anymore in the presence of process noise that is not negligible. By exploiting the results from the preceding section, the following simple example, however, constitutes a case where common process noise can consistently be treated by ICI.

We consider five sensor nodes deployed to observe a dynamic process, as depicted in Fig. 4. The state is two-dimensional, and its evolution is modeled by

$$
\underline{\mathbf{x}}_{k+1} = \mathbf{A}_k\,\underline{\mathbf{x}}_k + \underline{\mathbf{w}}_k
$$

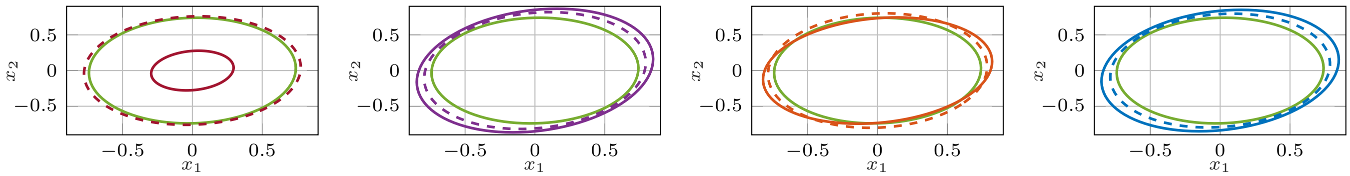
with system matrix $\mathbf{A}_k = \left[\begin{smallmatrix} 1 & 0.5 \\ 0 & 1 \end{smallmatrix}\right]$ and zero-mean process noise $\underline{\mathbf{w}}_k \sim \mathcal{N}(\underline{0}, \mathbf{C}^w)$. The process noise has the covariance matrix $\mathbf{C}^w = 0.5\,\mathbf{I}_2$. Each sensor node computes an estimate by means of a local Kalman filter. The filter processes measurements over five time steps before the local estimates are sent to node S5 according to the communication scheme shown in Fig. 4. Each node employs different fusion algorithms to combine the received estimates with its own estimate. The sensor nodes directly observe the state according to

$$
\underline{\mathbf{z}}_k = \mathbf{H}_k^{\mathsf{Si}}\,\underline{\mathbf{x}}_k + \underline{\mathbf{v}}_k^{\mathsf{Si}}
$$

with $\mathbf{H}_k^{\mathsf{Si}} = \mathbf{I}_2$ for $\mathsf{S1}, \ldots, \mathsf{S5}$. The error covariance matrix of $\underline{\mathbf{v}}_k^{\mathsf{Si}} \sim \mathcal{N}(\underline{0}, \mathbf{C}^{v,\mathsf{Si}})$ is $\mathbf{C}^{v,\mathsf{Si}} = \left[\begin{smallmatrix} 0.5 & 0 \\ 0 & 0.2 \end{smallmatrix}\right]$ for S1, S3, S5 and $\mathbf{C}_k^{v,\mathsf{Si}} = \left[\begin{smallmatrix} 0.1 & 0 \\ 0 & 0.5 \end{smallmatrix}\right]$ for S2, S4. The prior estimate has zero mean and the covariance matrix $\mathbf{C}_0^{\mathsf{Si}} = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$.

In order to compare the results of different fusion methods, 100 000 Monte Carlo runs have been performed. They are used to compute the actual error covariance matrix $\mathrm{E}[(\underline{\mathbf{x}} - \underline{\hat{\mathbf{x}}})(\underline{\mathbf{x}} - \underline{\hat{\mathbf{x}}})^{\mathrm{T}}]$ for each fusion method. In this scenario, both common process noise and common information are responsible for correlations between the estimates to be fused. Fig. 5 shows that a naïve fusion fails severely at providing a consistent estimate. CI provides a consistent estimate, while EI does not preserve consistency. The ICI fusion rule achieves a slightly better performance than CI; it still holds $\mathbf{C}_{\mathrm{CI}} > \mathbf{C}_{\mathrm{ICI}}$—although barely visible—and also the actual error has decreased, i.e., it holds $\mathrm{E}[\underline{\tilde{\mathbf{x}}}_{\mathrm{CI}}^{\mathrm{T}}\underline{\tilde{\mathbf{x}}}_{\mathrm{CI}}] > \mathrm{E}[\underline{\tilde{\mathbf{x}}}_{\mathrm{ICI}}^{\mathrm{T}}\underline{\tilde{\mathbf{x}}}_{\mathrm{ICI}}]$.

Compared with the example in Sec. II-C, where ICI provides a far smaller covariance matrix, as it can be seen in Fig. 1, ICI only slightly outperforms CI here. The key message of this



(a) Naïve fusion result. (b) CI fusion result. (c) EI fusion result. (d) ICI fusion result.

Fig. 5: Fusion results in node S5 at the end of the communication path in Fig. 4. The dashed ellipsoid represents the actual error covariance matrix, which is compared with the reported covariance matrix shown as solid ellipsoid. The green ellipsoid corresponds to the covariance matrix of the best attainable estimate provided by a centralized Kalman filter.

section, however, is not about the performance of the fusion methods; the key message is that ICI is consistent, and we can even confirm this assertion with the aid of the considerations in Sec. III. In particular, the treatment of common process noise has to be considered. The noise parameters in this example adhere to the inequalities

$$\mathbf{C}^w \geq \mathbf{C}^{v,\mathsf{Si}} \quad \text{for every } \mathsf{S1}, \ldots, \mathsf{S5}. \tag{26}$$

For two sensor nodes Si and Sj, we use the abbreviations $\mathbf{C}^{\mathrm{p}}_{\mathsf{A}} := \mathbf{C}^{\mathsf{Si}}_{k|k-1}$ and $\mathbf{C}^{\mathrm{p}}_{\mathsf{B}} := \mathbf{C}^{\mathsf{Sj}}_{k|k-1}$ for the predicted error covariance matrices of the estimates $\hat{\underline{\mathbf{x}}}^{\mathsf{Si}}_{k|k-1}$ and $\hat{\underline{\mathbf{x}}}^{\mathsf{Sj}}_{k|k-1}$, respectively, and $\mathbf{C}_{\mathsf{A}} := \mathbf{C}^{\mathsf{Si}}_{k|k}$ and $\mathbf{C}_{\mathsf{B}} := \mathbf{C}^{\mathsf{Sj}}_{k|k}$ for the corresponding covariance matrices after the measurement update step. In the prediction from time step $k-1$ to k, the covariance matrices

$$\mathbf{C}^{\mathrm{p}}_{\mathsf{A}} = \mathbf{A}_{k-1}\mathbf{C}^{\mathsf{Si}}_{k-1|k-1}\mathbf{A}^{\mathrm{T}}_{k-1} + \mathbf{C}^w \ , \tag{27a}$$

$$\mathbf{C}^{\mathrm{p}}_{\mathsf{B}} = \mathbf{A}_{k-1}\mathbf{C}^{\mathsf{Sj}}_{k-1|k-1}\mathbf{A}^{\mathrm{T}}_{k-1} + \mathbf{C}^w \tag{27b}$$

are computed. This leads to the inequalities $\mathbf{C}^{\mathrm{p}}_{\mathsf{A}} \geq \mathbf{C}^w$ and $\mathbf{C}^{\mathrm{p}}_{\mathsf{B}} \geq \mathbf{C}^w$. At this point, correlations between the predicted estimation errors can be arbitrary and, in particular, the process noise $\underline{\mathbf{w}}_k$ considered by each local Kalman filter introduces further correlations—known as the problem of common process noise. In the filtering step, we obtain

$$\left(\mathbf{C}_{\mathsf{A}}\right)^{-1} = \left(\mathbf{C}^{\mathrm{p}}_{\mathsf{A}}\right)^{-1} + \left(\mathbf{C}^{v,\mathsf{Si}}\right)^{-1} \ , \tag{28a}$$

$$\left(\mathbf{C}_{\mathsf{B}}\right)^{-1} = \left(\mathbf{C}^{\mathrm{p}}_{\mathsf{B}}\right)^{-1} + \left(\mathbf{C}^{v,\mathsf{Sj}}\right)^{-1} \tag{28b}$$

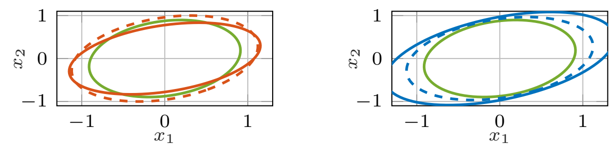
due to the identity measurement model, which implies $\left(\mathbf{C}^{v,\mathsf{Si}}\right)^{-1} \leq \left(\mathbf{C}_{\mathsf{A}}\right)^{-1}$ and $\left(\mathbf{C}^{v,\mathsf{Sj}}\right)^{-1} \leq \left(\mathbf{C}_{\mathsf{B}}\right)^{-1}$. We arrive at the inequalities

$$\mathbf{\Gamma}^{-1}_{\mathsf{A}} \leq \left(\mathbf{C}^{\mathrm{p}}_{\mathsf{A}}\right)^{-1} \overset{\text{(27a)}}{\leq} \left(\mathbf{C}^w\right)^{-1} \overset{\text{(26)}}{\leq} \left(\mathbf{C}^{v,\mathsf{Sj}}\right)^{-1} \overset{\text{(28b)}}{\leq} \left(\mathbf{C}_{\mathsf{B}}\right)^{-1}$$

$$\mathbf{\Gamma}^{-1}_{\mathsf{B}} \leq \left(\mathbf{C}^{\mathrm{p}}_{\mathsf{B}}\right)^{-1} \overset{\text{(27b)}}{\leq} \left(\mathbf{C}^w\right)^{-1} \overset{\text{(26)}}{\leq} \left(\mathbf{C}^{v,\mathsf{Si}}\right)^{-1} \overset{\text{(28a)}}{\leq} \left(\mathbf{C}_{\mathsf{A}}\right)^{-1} \ ,$$

where all possibly correlated parts are related to the predicted covariance matrix on the left-hand side since the measurement noise terms $\underline{\mathbf{v}}^{\mathsf{Si}}_k$ and $\underline{\mathbf{v}}^{\mathsf{Sj}}_k$ are uncorrelated. Hence, condition (21) is fulfilled if fusion takes place after a measurement update; ICI yields a consistent estimate.

The example demonstrates that the problem of common process noise can be treated by ICI. It is an interesting observation in (26) that the common process noise is even larger than the measurement noise, which is surprising if we note that ICI has been designed for unknown common information. We assume that ICI can be also applied to more general and extended fusion problems with proven consistency. Fig. 6 illustrates results for the same network, where the sensors S1, S3, S5 only measure the first component of the state and sensors S2 and S4 only the second component with variance $\mathbf{C}^{v,\mathsf{Si}} = 1$. Inequality (26) does not hold in this case, but ICI still provides consistent results. More general conditions for consistency will hence be subject of future work.



(a) EI fusion result. (b) ICI fusion result.

Fig. 6: Fusion results in node S5 for different sensors.

V. Conclusion

For decentralized data fusion, methods are desired that can cope with unknown correlations between the estimates to be fused. Consistent fusion rules that provide less conservative error covariance matrices than CI are only attainable if specific requirements on the correlation structure are met. An often encountered example is common information that is shared by the estimates and must not be double counted. ICI has been tailored to the treatment of unknown common information and therefore provides an optimal, i.e., tight, fusion result. Consequently, ICI reports a smaller error than CI. This paper demonstrates that ICI can also be applied to fusion problems, where other causes of correlations are present. In particular, the conditions under which ICI provides consistent fusion results have been relaxed, and it turns out that ICI can even be employed to tackle fusion problems where common process noise comes into play. These findings may indicate that ICI has the potential to become a viable fusion rule for typical Kalman filter-based fusion problems in general. Therefore, the relaxed consistency conditions stimulate further research on the field of fusion problems that can be covered by ICI. A second interesting aspect concerns nonlinear decentralized estimation, and it is to be studied whether ICI can be generalized to fuse probability densities that share common information as CI has been generalized in terms of exponential mixture densities.

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