Nonlinear Progressive Filtering for $SE(2)$ Estimation

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***Abstract*—In this paper, we present a novel nonlinear progressive filtering approach for estimating $SE(2)$ states represented by unit dual quaternions. Unlike previously published approaches, the measurement model no longer needs to be assumed as identity. Our solution utilizes deterministic sampling on a Bingham-like probability distribution, which has been adapted to simultaneously model orientation and translation. During the measurement update step, the estimate gets progressively updated. Our approach inherently incorporates the nonlinear structure of $SE(2)$ and enables a flexible measurement update step. We also give an evaluation for planar rigid body motion estimation with a case study that is close to real-world scenarios.**

***Keywords*—Pose estimation, Unit dual quaternion, Directional statistics, Nonlinear system, Progressive update, Bayesian filtering**

I. Introduction

Rigid body motions incorporate both orientation and position, they are ubiquitous, and their estimation is of significant importance in a variety of application scenarios, including virtual reality, augmented reality as well as robotic manipulation, navigation, and perception tasks. Mathematically speaking, any rigid body motion belongs to the special Euclidean group $SE(2)$ for the two-dimensional and $SE(3)$ for the three-dimensional case, respectively. Due to the nonlinear structure of its domain and correlation between orientation and translation, estimation using conventional stochastic filtering methods is challenging.

In order to resolve the issue of nonlinearity, some filtering algorithms focus on estimating uncertain pose on a locally linear domain of the nonlinear manifold, e.g., the well-known Extended Kalman Filter (EKF) and Unscented Kalman Filter (UKF) [1]. However, these can only give valid estimates in the case of sufficiently low noise where the nonlinear manifold can be reasonably well approximated with a linear subspace. In practical scenarios where noise level is relatively large due to limited hardware performance and particularly with fast movement, conventional approaches may lose tracking. In this case, additional sensors may need to be employed and sensor fusion is therefore necessary [2], [3], which can also enlarge the system and power burden. Furthermore, conventional approaches are always based on the assumption that the noise is Gaussian-distributed, which neglects its periodic nature and does not provide a proper probabilistic interpretation for the nonlinear structure of the underlying manifold.

Directional statistics, a subfield of statistics, particularly handles uncertain directional variables that are inherently defined on nonlinear domains. Some specific directional probability distributions such as the von Mises Distribution, the (Bivariate) Wrapped Normal Distribution or the Bingham

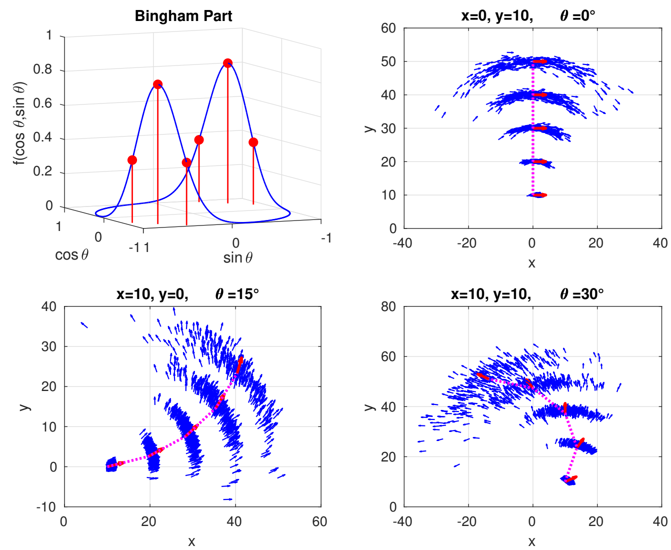

Figure 1: In this paper, dual quaternions are used to represent $SE(2)$ states and their uncertainty is modeled by the distribution proposed in [4], in which the orientational part is assumed to be Bingham-distributed. An example of the Bingham distribution on $\mathbb{S}^1$ is depicted in the top-left figure where samples are drawn deterministically. The other three pictures give examples of how the uncertainty modeled by the proposed distribution gets propagated through different system dynamics. For each propagation step, we randomly sample from the proposed distribution with a parameter matrix of $\mathbf{C} = -\mathrm{diag}(1, 200, 200, 200)$. The red arrow indicates the mode of the state distribution after each step of movement.

Distribution [5] have recently been used for performing recursive estimation of angular data [6], [7] and orientations [8], [9]. These methods typically rely on sampling on nonlinear manifolds and use a more intelligent scheme for propagation (e.g., unscented transform) through a nonlinear system. For handling nonlinearity and state spaces of arbitrary dimension, deterministic sampling is used. Unlike other schemes such as the Particle Filter (PF), which performs random Monte Carlo sampling, a deterministic method gives repeatable results, bypasses the particle degeneration issue, and is more computationally economical in the high-dimensional case.

In order to estimate orientation and position simultaneously, a dual quaternion-based Bayesian filtering approach has been introduced in [10]. In [11], the Partially-Conditioned Gaussian Distribution is developed, but it lacks a probabilistic interpretation for the correlation between translation and rotation term.

In [12], dual quaternions representing 6-DOF poses are modeled by the so-called Projected Gaussian, which is constructed by selecting a tangent point on the manifold of rotational quaternions $\mathbb{S}^3$ and projecting a higher-dimensional Gaussian distribution to $\mathbb{S}^3$ in an intelligent way to induce a distribution over 6D poses. However, it uses the Jacobian of the nonlinear transforms to estimate covariances as approximation. In [13], [14], Gaussian distributions are employed on the tangent planes to model uncertain dual quaternions on $SE(3)$ manifold, which lack probabilistic interpretation of the nonlinear manifold itself.

In [4], a Bingham-like distribution has been proposed to directly model uncertain dual quaternions on the manifold of $SE(2)$. It uses a Bingham distribution to model the real part (orientational quaternion) and a Gaussian distribution for the dual part (translational quaternion) conditioned on the orientation. Fig. 1 further visualizes the Bingham part of this distribution and shows the propagation of uncertain poses modeled by the proposed distribution through different system dynamics. The distribution considers the inherently nonlinear structure of $SE(2)$, gives probabilistic interpretation for the correlation between uncertain orientations and positions, and avoids errors resulting from improper linearization. Based on this proposed distribution, a UKF-based Bayesian filter is further introduced in [15]. It uses an approach similar to the unscented transform based on deterministic sampling for system propagation and a closed-form update step for measurement fusion. One of the main drawbacks of this filter is that the measurement model is restricted to be the identity function, which could be misleading because normally sensor data is not given in dual quaternion form with noise following the proposed distribution. The identity measurement assumption also makes it inconvenient to fuse instantly available sensor data directly and may require additional operations such as sensor synchronization to collect all the required data for getting a measurement of the same domain as the system state. Some other dual quaternion-based filtering schemes need multiple simultaneous measurements to rewrite the measurement model in linear form [16], which also limits its performance in practical use.

In this paper, we introduce a novel nonlinear recursive Bayesian filter for the estimation of $SE(2)$ states, namely the planar motions, with both system dynamics and measurement model arbitrarily given. This is achieved by using a dual quaternion as the representation for the $SE(2)$ state, which is modeled by a Bingham-like distribution to capture its nonlinear structure. The filtering procedure is performed as follows. First, we sample deterministically from the distribution and propagate the samples through the system function in the prediction step to get the prior. Second, during the update step, we use a novel progressive method based on Bayesian inference with an arbitrarily given likelihood to correct the prior given the measurement. Our approach considers the nonlinear structure of $SE(2)$, gives better resulting estimates in the case of fast motion and large noise, and no longer restricts system and measurement models to be identity functions, which extends its usage for nonlinear cases.

In Sec. II, preliminaries about dual quaternion manipulation for planar rigid body motions are introduced followed by an in-depth discussion of the probabilistic distribution we use in Sec. III. In Sec. IV, we show the components of our recursive estimation scheme and give a detailed introduction to the novel progressive update approach. Furthermore, we give an evaluation in Sec. V on the performance of our proposed filter based on the simulation of a two-dimensional robot. The work is concluded in Sec. VI.

II. Background

The concept of quaternions was first introduced by William Rowan Hamilton [17]. Unit quaternions, namely quaternions of unit length, provide a natural way of representing spatial rotations belonging to the special orthogonal group $SO(3)$. Compared to other conventional representation methods such as Euler angles and homogeneous transformation matrix, it can avoid gimbal lock, is more intuitive to compose, and is numerically more stable. In order to represent spatial transformations that incorporate both rotation and translation simultaneously, the dual number theory introduced by Clifford is combined with Hamilton's quaternion theory and provides a similar manipulation approach as quaternions do for rotation. Some fundamental preliminaries for quaternions and dual quaternions can be found in [17]–[19]. In this section, we focus on manipulation rules for quaternions and dual quaternions in the context of planar rigid body motions.

A. Quaternions

A quaternion is defined to be the sum of the real part and three imaginary parts formulated as

$$\mathbf{q} = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}\,, \tag{1}$$

with $q_1, q_2, q_3, q_4 \in \mathbb{R}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denoting the quaternion units with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Meanwhile, the set of quaternions $\mathbb{H}$ can be seen as 4-dimensional vector space over the real numbers, i.e., $\mathbf{q} \in \mathbb{R}^4$. Arithmetics on $\mathbb{H}$ covers addition, scalar multiplication, conjugation, inverse and product, where the first two operations follow the same rule as ordinary vector arithmetics. The conjugate of $\mathbf{q}$ is defined as $\mathbf{q}^* = \operatorname{diag}(1, -1, -1, -1) \cdot \mathbf{q}$, whereas the inverse follows $\mathbf{q}^{-1} = \mathbf{q}^* / \left|\left|\mathbf{q}\right|\right|^2$, with $\left|\left|\mathbf{q}\right|\right|^2 = \mathbf{q} \otimes \mathbf{q}^*$. Here $\otimes$ denotes the Hamilton product. For $\mathbf{q} \in \mathbb{H}$, its norm equals its length in $\mathbb{R}^4$. The Hamilton product, which is noncommutative, can be also performed with matrix-vector multiplication, namely $\mathbf{p} \otimes \mathbf{q} = \mathbf{Q}_p^l \mathbf{q} = \mathbf{Q}_q^r \mathbf{p}$, with

$$\begin{aligned} \mathbf{Q}_p^l &= \begin{pmatrix} p_1 & -p_2 & -p_3 & -p_4 \\ p_2 & p_1 & -p_4 & p_3 \\ p_3 & p_4 & p_1 & -p_2 \\ p_4 & -p_3 & p_2 & p_1 \end{pmatrix}, \\ \mathbf{Q}_q^r &= \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & q_4 & -q_3 \\ q_3 & -q_4 & q_1 & q_2 \\ q_4 & q_3 & -q_2 & q_1 \end{pmatrix}. \end{aligned} \tag{2}$$

B. Unit Quaternions and Spatial Rotation

Quaternions of unit length are called unit quaternions. The set of unit quaternions $\mathbb{H}^1$ in Euclidean space further forms the manifold of a unit hypersphere $\mathbb{S}^3$, i.e., $\left\{\mathbf{q} \,|\, \mathbf{q} \in \mathbb{R}^4,\ \left|\left|\mathbf{q}\right|\right| = 1\right\}$. For representing rotations, a unit quaternion can be intuitively defined as

$$\mathbf{q} = \cos\left(\frac{\theta}{2}\right) + \mathbf{u}\sin\left(\frac{\theta}{2}\right)\,, \tag{3}$$

where θ and unit vector $\mathbf{u}$ denote rotation angle and rotation axis respectively. Two antipodal points on $\mathbb{S}^3$, e.g., $\mathbf{q}$ and $-\mathbf{q}$, denote the same rotation. Given a unit quaternion $\mathbf{q} \in \mathbb{H}^1$ defined as in (3), a point $\mathbf{v} \in \mathbb{R}^3$ can be rotated to $\mathbf{v}'$ according to $\mathbf{v}' = \mathbf{q} \otimes \mathbf{v} \otimes \mathbf{q}^*$.

C. Dual Quaternions

A dual quaternion is essentially an ordered pair of quaternions $(\mathbf{x}_r, \mathbf{x}_d) \in \mathbb{R}^8$. It is defined as

$$\mathbf{x} = \mathbf{x}_r + \epsilon\, \mathbf{x}_d \,, \tag{4}$$

with $\epsilon\,(\epsilon^2 = 0)$ denoting the *dual unit* and $\mathbf{x}_r$, $\mathbf{x}_d$ the *real* and *dual* parts respectively. Arithmetics defined on the set of dual quaternions $\mathbb{H}_d$ is therefore a combination of quaternion and dual-number theory. General rules of dual quaternion arithmetics can be found in [20]. There are overall three kinds of conjugates defined for dual quaternions, i.e., the *dual conjugate*, defined as the conjugate of the dual unit

$$\mathbf{x}^\bullet = \mathrm{diag}(1,1,1,1,-1,-1,-1,-1) \cdot \mathbf{x} \,,$$

the *classical conjugate*, which only conjugates each individual quaternion

$$\mathbf{x}^* = \mathrm{diag}(1,-1,-1,-1,1,-1,-1,-1) \cdot \mathbf{x} \,,$$

and the *full conjugate*, which is the combination of the former two conjugations

$$\mathbf{x}^\circ = (\mathbf{x}^\bullet)^* = \mathrm{diag}(1,-1,-1,-1,-1,1,1,1) \cdot \mathbf{x} \,.$$

The product of two dual quaternions can also be performed through matrix-vector multiplication, namely

$$\mathbf{x}_1 \otimes \mathbf{x}_2 = \mathbf{Q}^l_{x_1} \mathbf{x}_2 = \mathbf{Q}^r_{x_2} \mathbf{x}_1 \,,$$

with

$$\mathbf{Q}^l_{x_1} = \begin{pmatrix} \mathbf{Q}^l_{r_1} & \mathbf{0} \\ \mathbf{Q}^l_{d_1} & \mathbf{Q}^l_{r_1} \end{pmatrix} , \; \mathbf{Q}^r_{x_2} = \begin{pmatrix} \mathbf{Q}^r_{r_2} & \mathbf{0} \\ \mathbf{Q}^r_{d_2} & \mathbf{Q}^r_{r_2} \end{pmatrix} .$$

Here the submatrices are defined as in (2). Moreover, the norm of a dual quaternion is defined as

$$||\mathbf{x}|| = \sqrt{\mathbf{x} \otimes \mathbf{x}^*} \,,$$

which uses the classical conjugate.

D. Unit Dual Quaternions and Planar Motion

In this section, we only consider unit dual quaternions representing planar transformations, namely the group of $SE(2)$. Without loss of generality, the planar motion can be assumed to be in (x, y)-coordinates with rotation around z-axis and translation of $\mathbf{t} = [t_x, t_y]^T$. We can therefore represent the planar rotation with a unit quaternion

$$\mathbf{x}_r = \cos\left(\frac{\theta}{2}\right) + \mathbf{k} \sin\left(\frac{\theta}{2}\right) , \tag{5}$$

with $\mathbf{k}$ denoting the unit vector on z-axis. This can also be viewed as a two-dimensional vector $\mathbf{x}_r = [x_{r,1}, x_{r,2}]^T \in \mathbb{S}^1$ (unit circle). The dual part hereby can be defined as

$$\mathbf{x}_d = \frac{1}{2}\mathbf{t} \otimes \mathbf{x}_r = \frac{1}{2}\mathbf{Q}_r \mathbf{t} \,. \tag{6}$$

Matrix $\mathbf{Q}_r$ is essentially a degenerate case of (2) and can be derived as

$$\mathbf{Q}_r = \begin{pmatrix} x_{r,1} & x_{r,2} \\ -x_{r,2} & x_{r,1} \end{pmatrix} . \tag{7}$$

Since $\mathbf{x}_r \in \mathbb{S}^1$, it can be proven that $\mathbf{Q}_r \mathbf{Q}_r^T = \mathbf{Q}_r^T \mathbf{Q}_r = \mathbf{I}_{2\times 2}$ and $\det(\mathbf{Q}_r) = 1$, thus $\mathbf{Q}_r$ belongs to the two-dimensional rotation matrix group, i.e., $\mathbf{Q}_r \in SO(2)$. With real and dual part defined as in (5) and (6) individually, the dual quaternion is inherently guaranteed to be of unit length, namely

$$\begin{aligned} \mathbf{x} \otimes \mathbf{x}^* &= (\mathbf{x}_r + \epsilon\, \mathbf{x}_d) \otimes (\mathbf{x}_r^* + \epsilon\, \mathbf{x}_d^*) \\ &= \mathbf{x}_r \otimes \mathbf{x}_r^* + \epsilon\,(\mathbf{x}_r \otimes \mathbf{x}_d^* + \mathbf{x}_d \otimes \mathbf{x}_r^*) \\ &= \mathbf{x}_r \otimes \mathbf{x}_r^* + \frac{\epsilon}{2}(\mathbf{x}_r \otimes \mathbf{x}_r^* \otimes \mathbf{t}^* + \mathbf{t} \otimes \mathbf{x}_r \otimes \mathbf{x}_r^*) \\ &= \mathbf{x}_r \otimes \mathbf{x}_r^* + \frac{\epsilon}{2}(\mathbf{t}^* + \mathbf{t}) \\ &= \mathbf{x}_r \otimes \mathbf{x}_r^* = 1 \,. \end{aligned}$$

Planar transformations using unit dual quaternions can then be performed in a similar manner as three-dimensional rotations using quaternions. For example, we can transform a point $\mathbf{v}$ to $\mathbf{v}'$ with a unit dual quaternion $\mathbf{x}$ through

$$\begin{aligned} \mathbf{v}' &= \mathbf{x} \otimes \mathbf{v} \otimes \mathbf{x}^\circ \\ &= (\mathbf{x}_r + \frac{\epsilon}{2}\mathbf{t} \otimes \mathbf{x}_r) \otimes (1 + \epsilon \mathbf{v}) \otimes (\mathbf{x}_r + \frac{\epsilon}{2}\mathbf{t} \otimes \mathbf{x}_r)^\circ \\ &= (\mathbf{x}_r + \epsilon\, \mathbf{x}_r \otimes \mathbf{v} + \frac{\epsilon}{2}\mathbf{t} \otimes \mathbf{x}_r) \otimes (\mathbf{x}_r^* - \epsilon\, \frac{1}{2} \otimes \mathbf{x}_r^* \otimes \mathbf{t}^*) \\ &= \mathbf{x}_r \otimes \mathbf{x}_r^* + \epsilon\, \mathbf{x}_r \otimes \mathbf{v} \otimes \mathbf{x}_r^* + \\ &\qquad \frac{\epsilon}{2}(\mathbf{t} \otimes \mathbf{x}_r \otimes \mathbf{x}_r^* - \mathbf{x}_r \otimes \mathbf{x}_r^* \otimes \mathbf{t}^*) \\ &= 1 + \epsilon\,(\mathbf{x}_r \otimes \mathbf{v} \otimes \mathbf{x}_r^* + \mathbf{t}) \,, \end{aligned}$$

which translates a point $\mathbf{v}$ after rotating it. Multiplication of two dual quaternions representing planar motion can also be performed using matrix-vector multiplication $\mathbf{x}_1 \otimes \mathbf{x}_2 = \mathbf{Q}_1^l \mathbf{x}_2 = \mathbf{Q}_2^r \mathbf{x}_1$, with $\mathbf{x}_1, \mathbf{x}_2$ their vector form obtained by concatenating the two parts into one four-dimensional vector

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_r \\ \mathbf{x}_d \end{bmatrix} \in \mathbb{S}^1 \times \mathbb{R}^2 \,, \tag{8}$$

and

$$\mathbf{Q}_1^l = \begin{pmatrix} x_{1,1} & -x_{1,2} & 0 & 0 \\ x_{1,2} & x_{1,1} & 0 & 0 \\ x_{1,3} & x_{1,4} & x_{1,1} & -x_{1,2} \\ x_{1,4} & -x_{1,3} & x_{1,2} & x_{1,1} \end{pmatrix} , \tag{9}$$

$$\mathbf{Q}_2^r = \begin{pmatrix} x_{2,1} & -x_{2,2} & 0 & 0 \\ x_{2,2} & x_{2,1} & 0 & 0 \\ x_{2,3} & -x_{2,4} & x_{2,1} & x_{2,2} \\ x_{2,4} & x_{2,3} & -x_{2,2} & x_{2,1} \end{pmatrix} . \tag{10}$$

III. Stochastic Modeling of Uncertain Dual Quaternions for Planar Motion

Due to the fact that two unit dual quaternions $\mathbf{x}$ and $-\mathbf{x}$ represent the same rigid body motion, a density function modeling uncertain dual quaternions should be antipodally symmetric, namely $f(\mathbf{x}) = f(-\mathbf{x})$. Meanwhile, the density function marginalized on the orientation parts should also have the same property since two antipodal unit quaternions represent the same rotation. To address this issue, we first derive the joint probability density function for orientation and position parts of uncertain dual quaternions and further use it to interpret our previously proposed distribution in [4].

A. Joint Probability Density for Real and Dual Parts

The manifold of dual quaternions for planar motion is essentially the Cartesian product of the unit circle and two-dimensional Euclidean space, namely $\mathbb{S}^1 \times \mathbb{R}^2$, which is embedded in the four-dimensional Euclidean space $\mathbb{R}^4$. Using chain rule and Bayes' theorem, we can condition the density function on the rotational part, i.e., the real part of the dual quaternion, as follows

$$\begin{aligned} f(\mathbf{x}_r, \mathbf{x}_d) &= f(\mathbf{x}_d|\mathbf{x}_r) f(\mathbf{x}_r) \\ &= f(\mathbf{x}_r) \int_{\mathbb{R}^2} f(\mathbf{x}_d, \mathbf{t}|\mathbf{x}_r) d\mathbf{t} \\ &= f(\mathbf{x}_r) \int_{\mathbb{R}^2} f(\mathbf{x}_d|\mathbf{t}, \mathbf{x}_r) f(\mathbf{t}|\mathbf{x}_r) d\mathbf{t} \\ &= f(\mathbf{x}_r) \int_{\mathbb{R}^2} \delta(\mathbf{t} - 2\mathbf{x}_d \otimes \mathbf{x}_r^{-1}) f(\mathbf{t}|\mathbf{x}_r) d\mathbf{t} \\ &= f_{\mathbf{x}_r}(\mathbf{x}_r) f_{\mathbf{t}|\mathbf{x}_r}(2\mathbf{x}_d \otimes \mathbf{x}_r^{-1}) \,, \end{aligned}$$

with $\delta(\cdot, \cdot)$ denoting the Dirac delta distribution [21]. We further assume the orientation part to be Bingham-distributed [22] because of its antipodal symmetry property on the unit circle and use a bivariate Gaussian to model the uncertain position term $\mathbf{t}$ conditioned on the orientation part $\mathbf{x}_r$. Given the assumption $\mathbf{t}|\mathbf{x}_r \sim \mathcal{N}(\mu_{\mathbf{t}|\mathbf{x}_r}, \Sigma_{\mathbf{t}|\mathbf{x}_r})$ and the affine transformation in (6), the dual part conditioned on orientation should also follow a Gaussian distribution with transformed mean and covariance, i.e., $\mathbf{x}_d|\mathbf{x}_r \sim \mathcal{N}(\mu_{\mathbf{x}_d|\mathbf{x}_r}, \Sigma_{\mathbf{x}_d|\mathbf{x}_r})$, with

$$\begin{aligned} \mu_{\mathbf{x}_d|\mathbf{x}_r} &= \frac{1}{2} \mathbf{Q}_r \mu_{\mathbf{t}|\mathbf{x}_r} \,, \\ \Sigma_{\mathbf{x}_d|\mathbf{x}_r} &= \frac{1}{4} \mathbf{Q}_r \Sigma_{\mathbf{t}|\mathbf{x}_r} \mathbf{Q}_r^T \,. \end{aligned}$$

Thus, the conditional distribution can be written explicitly as

$$\begin{aligned} f(\mathbf{x}_d|\mathbf{x}_r) &= \frac{1}{\pi\sqrt{\det(\mathbf{Q}_r \Sigma_{\mathbf{t}|\mathbf{x}_r} \mathbf{Q}_r^T)}} \exp \\ &\left((\mathbf{x}_d - \frac{1}{2}\mathbf{Q}_r \mu_{\mathbf{t}|\mathbf{x}_r})^T (-2\mathbf{Q}_r \Sigma_{\mathbf{t}|\mathbf{x}_r}^{-1} \mathbf{Q}_r^T)(\mathbf{x}_d - \frac{1}{2}\mathbf{Q}_r \mu_{\mathbf{t}|\mathbf{x}_r}) \right) \\ &= \frac{1}{\pi\sqrt{\det(\Sigma_{\mathbf{t}|\mathbf{x}_r})}} \exp \\ &\left((\mathbf{x}_d - \frac{1}{2}\mathbf{Q}_r \mu_{\mathbf{t}|\mathbf{x}_r})^T (-2\mathbf{Q}_r \Sigma_{\mathbf{t}|\mathbf{x}_r}^{-1} \mathbf{Q}_r^T)(\mathbf{x}_d - \frac{1}{2}\mathbf{Q}_r \mu_{\mathbf{t}|\mathbf{x}_r}) \right) \,, \end{aligned}$$

where $\det(\mathbf{Q}_r) = 1$ since $\mathbf{Q}_r \in SO(2)$.

B. Distribution for Dual Quaternions of Planar Motion

We employ a distribution belonging to the exponential family that inherently respects the underlying structure of dual quaternions representing planar motions. The distribution, as proposed in [4], [15], is defined as

$$f(\mathbf{x}) = \frac{1}{N(\mathbf{C})} \exp(\mathbf{x}^T \mathbf{C} \mathbf{x}) \,, \tag{11}$$

which is the probability density distribution for a random dual quaternion $\mathbf{x} = [\mathbf{x}_r^T, \mathbf{x}_d^T]^T \in \mathbb{S}^1 \times \mathbb{R}^2$. The matrix $\mathbf{C} \in \mathbb{R}^{4\times4}$ is defined as

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2^T \\ \mathbf{C}_2 & \mathbf{C}_3 \end{pmatrix} \,, \tag{12}$$

with $\mathbf{C}_i \in \mathbb{R}^{2\times2}$, symmetric $\mathbf{C}_1$, arbitrary $\mathbf{C}_2$, and symmetric negative definite $\mathbf{C}_3$, guaranteeing the proposed probability density function to be well defined. It can be further decomposed by computing the Schur complement of $\mathbf{C}_3$ such that

$$\begin{aligned} &f(\mathbf{x}) \\ &= \frac{1}{N(\mathbf{C})} \exp(\mathbf{x}_r^T \mathbf{T}_1 \mathbf{x}_r + (\mathbf{x}_d - \mathbf{T}_2\mathbf{x}_r)^T \mathbf{C}_3 (\mathbf{x}_d - \mathbf{T}_2\mathbf{x}_r)) \\ &= f(\mathbf{x}_r) f(\mathbf{x}_d|\mathbf{x}_r) \,, \end{aligned}$$

with $\mathbf{T}_1 = \mathbf{C}_1 - \mathbf{C}_2^T \mathbf{C}_3^{-1} \mathbf{C}_2$ denoting the parameter matrix of the Bingham distribution modeling the orientation quaternion $\mathbf{x}_r$ and $\mathbf{C}_3$ the Gaussian parameter matrix modeling the position quaternion $\mathbf{x}_d$ conditioned on orientation. Here $\mathbf{T}_2 = \mathbf{C}_3^{-1}\mathbf{C}_2$ denotes the dependency between orientation and position parts. The normalization constant can thus be calculated as

$$N(\mathbf{C}) = \frac{2\pi\sqrt{\det(-0.5\mathbf{C}_3^{-1})}}{B(\mathbf{T}_1)} \,, \tag{13}$$

where $B(\mathbf{T}_1)$ denotes the normalization constant of the Bingham distribution determined by $\mathbf{T}_1$ [4, Sec. III]. The parameter matrix $\mathbf{C}_3$ indirectly determines the Gaussian distribution for a position term $\mathbf{t}$ conditioned on the orientation with covariance

$$\begin{aligned} \Sigma_{\mathbf{t}|\mathbf{x}_r} &= (-2\mathbf{Q}_r^T \mathbf{C}_3 \mathbf{Q}_r)^{-1} \\ &= -\frac{1}{2} \mathbf{Q}_r^T \mathbf{C}_3^{-1} \mathbf{Q}_r \,, \end{aligned} \tag{14}$$

and mean value

$$\mu_{\mathbf{t}|\mathbf{x}_r} = 2\mathbf{Q}_r^T \mathbf{T}_2 \mathbf{x}_r \,. \tag{15}$$

The matrix $\mathbf{Q}_r$ defines essentially a rotation matrix around the z-axis with rotation angle $-\theta/2$. Given the parameters of the proposed distribution, we can thus compute the distribution of $\mathbf{t}$ by performing a rotation.

IV. Progressive Bayesian Filtering Approach for Planar Motions

A typical recursive filtering algorithm comprises two steps for each individual time step. They are the prediction step, which propagates the system state with uncertainty according to the system dynamics model, and the update step, which fuses the uncertain measurement in a stochastic manner to correct the current estimate. The filter proposed in this paper uses dual quaternions to represent planar motions as the system state and assumes the following system dynamics

$$\mathbf{x}_{k+1} = a(\mathbf{x}_k, \mathbf{u}_k) \otimes \mathbf{w}_k \,, \tag{16}$$

with $\mathbf{x}_k, \mathbf{w}_k \in \mathbb{S}^1 \times \mathbb{R}^2$ representing the current system state and system noise respectively. For each recursive step, the robot pose gets transformed according to the current system input $\mathbf{u}_k$ and system dynamics $a(\cdot, \cdot)$ and then gets propagated with non-additive noise $\mathbf{w}_k$ that follows the proposed distribution in (11) parameterized by a matrix $\mathbf{C}_k^w$. Moreover, the measurement model is assumed to be non-identity as

$$\mathbf{z}_k = h(\mathbf{x}_k) \boxplus \mathbf{v}_k , \tag{17}$$

with $\mathbf{z}_k, \mathbf{v}_k$ denoting the current measurement and measurement noise, which are in our case not necessarily in the same domain as the system state $\mathbf{x}_k \in \mathbb{S}^1 \times \mathbb{R}^2$. Here we use $\boxplus$ for either additive or non-additive noise. The function $h(\cdot)$ maps the current dual quaternion state to the measurement domain and a noise term $\mathbf{v}_k$ depending on the individual noise model are applied. This brings mainly two advantages compared to the originally proposed identity noise model. First, the measurement noise can be better and more closely modeled for real-world scenarios without conversion to the domain of system state. Second, the measurement model can also be conveniently switched at each individual time step and measurements from different sensors can get fused depending on their instant availability. This is specifically suitable for the case of multi-sensor fusion, where, e.g., a robot is mounted with several sensors, even of different type, working in an unsynchronized manner. For each recursive step, the current state is represented by the proposed distribution, whereas the estimated pose can be derived as the distribution's mode as introduced in [4, Sec. III]. (15) and (14) also provide a convenient approach to measure and visualize estimation uncertainty of the position part.

A. Prediction

For the prediction step, we employ an approach similar to the one proposed in [15, Alg. 3] with extension to incorporate the system input. Here, weighted samples are first drawn from the last estimated state distribution characterized by $\mathbf{C}_{k-1}^e$ deterministically [15, Alg. 2]. Then, they are transformed according to the system input $\mathbf{u}_k$ and are further propagated with uncertain noise samples deterministically drawn from the noise distribution with parameter matrix $\mathbf{C}^w$. Finally, the estimated state is approximated by the parameter estimation approach introduced in [15, Alg. 1].

Algorithm 1 Prediction

procedure predict($\mathbf{C}_{k-1}^e, \mathbf{C}^w, \mathbf{u}_k$)
- $\{(\mathbf{x}_i^e, p_i^e)\}_{i=1,\cdots,n} \leftarrow$SampleDeterministic($\mathbf{C}_{k-1}^e$);
- $\{(\mathbf{x}_j^w, p_j^w)\}_{j=1,\cdots,n} \leftarrow$SampleDeterministic($\mathbf{C}^w$);
- $l \leftarrow 0$;
- **for** $i = 1$ to n **do**
  - **for** $j = 1$ to n **do**
    - $l \leftarrow l + 1$;
    - $\mathbf{s}_l \leftarrow a(\mathbf{x}_i^e, \mathbf{u}_k) \otimes \mathbf{x}_j^w$;
    - $p_l \leftarrow p_i^e \cdot p_j^w$;
  - **end for**
- **end for**
- **return** $\mathbf{C}_k^p \leftarrow$ EstimateParameters($\{(\mathbf{s}_l, p_l)\}_{l=1,\cdots,n^2}$);

end procedure

B. Nonlinear Progressive Update

According to Bayes' rule, in the update step we aim to correct the predicted state given the current measurement via

$$f(\mathbf{x}_k|\mathbf{z}_k) = \frac{f(\mathbf{z}_k|\mathbf{x}_k) \cdot f(\mathbf{x}_k)}{f(\mathbf{z}_k)} \propto f(\mathbf{z}_k|\mathbf{x}_k) \cdot f(\mathbf{x}_k) , \tag{18}$$

with $f(\mathbf{x}_k)$ denoting the prior and $f(\mathbf{z}_k|\mathbf{x}_k)$ the likelihood. The likelihood function is assumed to be explicitly given. Provided the operator $\boxplus$ is invertible, the likelihood can be derived according to

$$\begin{aligned} f(\mathbf{z}_k|\mathbf{x}_k) &= \int_Z f(\mathbf{z}_k, \mathbf{v}_k|\mathbf{x}_k) d\mathbf{v}_k \\ &= \int_Z f(\mathbf{z}_k|\mathbf{v}_k, \mathbf{x}_k) f(\mathbf{v}_k) d\mathbf{v}_k \\ &= \int_Z \delta(\mathbf{v}_k - (h(\mathbf{x}_k))^{-1} \boxplus \mathbf{z}_k) f(\mathbf{v}_k) d\mathbf{v}_k \\ &= f_{\mathbf{v}_k}((h(\mathbf{x}_k))^{-1} \boxplus \mathbf{z}_k) . \end{aligned} \tag{19}$$

For an identity measurement model with noise from the proposed distribution for dual quaternions, the likelihood can be directly computed as $f(\mathbf{z}_k|\mathbf{x}_k) = f_{\mathbf{v}_k}(\mathbf{x}_k^{-1} \otimes \mathbf{z}_k)$, with $\mathbf{v}_k$ following the Bingham-like distribution. This can be used to derive the closed-form solution for the update step as introduced in [15].

The product of the prior and the likelihood in (18) normally cannot be evaluated analytically. Instead, we can deterministically draw prior samples, multiply the sample weights componentwise with their corresponding likelihood given the measurement, and further re-approximate the newly weighted samples to the proposed distribution to get the posterior. However, this approach can inherently suffer from the problematic sample degeneration, namely samples after update can be assigned weights close to zero and cannot be approximated with a reasonable continuous density. This could be risky, especially for the approach where we use deterministic sampling, as e.g., for the domain of system state $\mathbb{S}^1 \times \mathbb{R}^2$, only 15 samples are drawn for the update step.

To resolve this problem, a progressive update approach is proposed for carrying out Bayesian inference reliably and more accurately during the update step for dual quaternions representing planar motion. This approach has been successfully used for both linear and nonlinear manifolds such as the Euclidean space $\mathbb{R}^n$ [23], unit circle $\mathbb{S}^1$ [24] and [25], and torus $\mathbb{S}^1 \times \mathbb{S}^1$ [26]. The basic philosophy in this approach is to progressively multiply likelihoods with the prior in which (18) can be converted into

$$\begin{aligned} f(\mathbf{x}_k|\mathbf{z}_k) &\propto f(\mathbf{z}_k|\mathbf{x}_k) f(\mathbf{x}_k) \\ &= f(\mathbf{z}_k|\mathbf{x}_k)^{\lambda_n} \cdots f(\mathbf{z}_k|\mathbf{x}_k)^{\lambda_1} f(\mathbf{x}_k) , \end{aligned} \tag{20}$$

with $\lambda_1, \cdots, \lambda_n \in (0, 1]$ and $\sum_{i=1}^n \lambda_i = 1$, which determine the progression step size and can be used to avoid sample degeneration. For each progression step i, we perform deterministic sampling for the prior to get a set of weighted samples. Given the current measurement $\mathbf{z}_k$, we then calculate their individual likelihoods and ensure the quotient of the smallest over the largest likelihood not to be smaller than a pre-defined threshold term $\tau \in (0, 1]$, i.e.,

$$\frac{s_{\min}}{s_{\max}} \geq \tau \in (0, 1] , \tag{21}$$

with weights s_j denoting the scaled likelihood for each prior sample $\mathbf{x}_k^j$, i.e.,

$$s_k^j = f(\mathbf{z}_k|\mathbf{x}_k^j)^{\lambda_i} = f_{\mathbf{v}_k}((h(\mathbf{x}_k^j))^{-1} \boxplus \mathbf{z}_k)^{\lambda_i} . \tag{22}$$

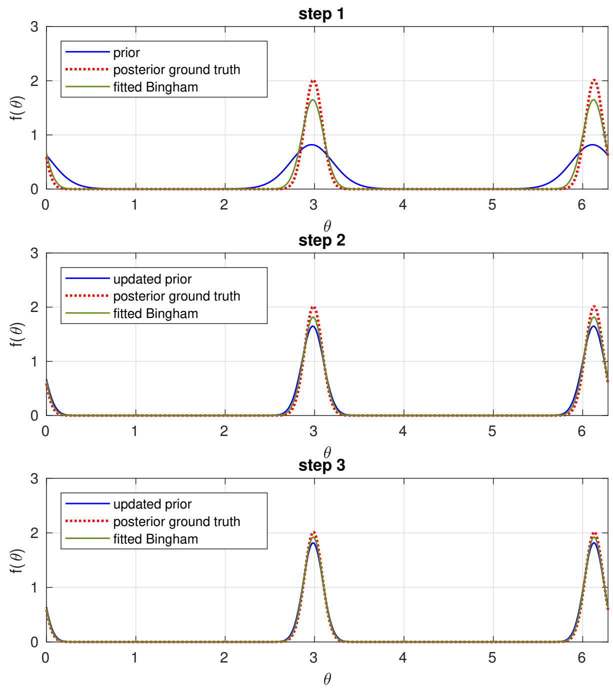

Figure 2: Example of the progressive update approach with $\tau = 0.3$. After each progression step, the Bingham part of the estimated posterior gets closer to the ground truth posterior distribution.

Here the prior samples $\{\mathbf{x}_k^j\}_{j=1,\cdots,m}$ are deterministically drawn. The progression step size can thus be computed according to

$$\frac{s_{\min}}{s_{\max}} = \frac{\min\limits_{j=1,\cdots,m} s_k^j}{\max\limits_{j=1,\cdots,m} s_k^j} = \frac{\min\limits_{j=1,\cdots,m} f_{\mathbf{v}_k}(\mathbf{z}_k|\mathbf{x}_k^j)^{\lambda_i}}{\max\limits_{j=1\cdots,m} f_{\mathbf{v}_k}(\mathbf{z}_k|\mathbf{x}_k^j)^{\lambda_i}} \geq \tau,$$

thus the scale term λ_i for current progression step can be further derived as

$$\lambda_i \leq \frac{\log(\tau)}{\log\left(\frac{\min\limits_{1,\cdots,m} f(\mathbf{z}_k|\mathbf{x}_k^j)}{\max\limits_{1,\cdots,m} f(\mathbf{z}_k|\mathbf{x}_k^j)}\right)}\,. \tag{23}$$

The likelihood can then get scaled by the resulting λ_i and multiplied with the weights of the prior samples. The estimated density can be re-approximated from updated samples and act as the new prior for next round of progression update. The detailed progressive update algorithm for generic measurement models is given in Alg. 2. Here, an error handling is added on line 10 in case of zero maximum likelihood during a certain progression step. Fig. 2 shows how the orientation part, which is a Bingham distribution, gets gradually updated after each progression step.

Algorithm 2 Nonlinear progressive update

procedure progressiveUpdate($\mathbf{z}_k, \mathbf{C}_k^p, \tau$)
1: $\Lambda \leftarrow 1$;
2: $i \leftarrow 0$;
3: $\mathbf{C}_k^e \leftarrow \mathbf{C}_k^p$;
4: **while** $\Lambda > 0$ **do**
5: $i \leftarrow i + 1$;
6: $\{(\mathbf{x}_j, p_j)\}_{j=1,\cdots,m} \leftarrow$SampleDeterministic($\mathbf{C}_k^e$);
7: $p_{\min} \leftarrow \min\limits_{j=1,\cdots,m} f(\mathbf{z}_k|\mathbf{x}_k^j)$;
8: $p_{\max} \leftarrow \max\limits_{j=1,\cdots,m} f(\mathbf{z}_k|\mathbf{x}_k^j)$;
9: **if** $p_{\max} = 0$ **then**
10: **return** $\mathbf{C}_k^e$;
11: **end if**
12: $\lambda_i \leftarrow \min(\Lambda, \frac{\log(\tau)}{\log(p_{\min}/p_{\max})})$;
13: **for** $j = 1$ **to** m **do**
14: $p_j \leftarrow f(\mathbf{z}_k|\mathbf{x}_k^j)^{\lambda_i} \cdot p_j$;
15: **end for**
16: $\mathbf{C}_k^e \leftarrow$ EstimateParameters($\{x_j, p_j\}_{j=1,\cdots,m}$);
17: $\Lambda \leftarrow \Lambda - \lambda_i$;
18: **end while**
19: **return** $\mathbf{C}_k^e$;
end procedure

V. Evaluation

In this part, we aim to compare our proposed filter to the filter originally proposed in [15] in a simulation scenario. We use a simulated "crawler", which is essentially a miniature walking robot, to perform planar motion in a structured and static map $\mathcal{M}$. The crawler is equipped with four range sensors (e.g. ultrasonic sensors) on each side of its body measuring distance from the obstacles (e.g., walls) and an IMU measuring the orientation as shown in Fig. 3. We assume that the ultrasonic sensor does not have a measurement cone but just a line along which it measures and further assume the IMU to be drift-free. The robot's dynamics model is given as follows

$$\mathbf{x}_{k+1} = \mathbf{x}_k \otimes \mathbf{u}_k \otimes \mathbf{w}_k\,,$$

with $\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k \in \mathbb{S}^1 \times \mathbb{R}^2$. The system state, namely position $\mathbf{t} = [x, y]^T$ and orientation θ, is represented using a dual quaternion as formulated in (8). The system input $\mathbf{u}_k$ is also given as dual quaternions representing stepwise planar movements and the dynamics noise is distributed according to the distribution in (11) with a parameter matrix

$$\mathbf{C}^w = \text{diag}(-1, -100, -100, -100)\,,$$

which has the mode on $[1, 0, 0, 0]^T$ indicating system noise to be zero-centered for both position and orientation. We further assume the noise of the four ultrasonic sensors as Gaussian-distributed in $\mathbb{R}^4$ with zero mean and a covariance matrix of $\Sigma_\mathbf{d} = \text{diag}(20, 20, 20, 20)$. Since the map information $\mathcal{M}$ is already given, the crawler's position can thus be computed by performing trilateration with distance measurements. Orientation measurements from the IMU are hereby assumed to be Bingham-distributed with a parameter matrix of $\mathbf{C}^{\text{ort}} = \text{diag}(0, -50)$, whose mode also indicates a zero-centered angular noise. The measurement model can therefore be formulated with respect to different filtering approaches as follows.

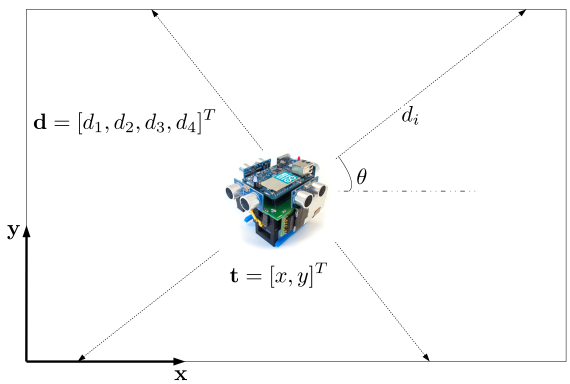

Figure 3: Crawler performing planar rigid body motion in a simulated scenario.

A. Non-identity Measurement Model

By employing our proposed progressive update approach, the measurement model no longer needs to be assumed as identity, but is given by

$$\mathbf{z}_k = h_{\mathcal{M}}(\mathbf{x}_k) + \mathbf{d}_k \,,$$

where the measurement $\mathbf{z}_k$ can consist of the direct distance measurements from the ultrasonic sensors and the function $h_{\mathcal{M}}(\cdot)$ maps the current state $\mathbf{x}_k$ to the domain $\mathbb{R}^4$ by doing ray casting given the map information $\mathcal{M}$. Here, range measurement noise is assumed to be an zero-centered four-dimensional Gaussian, namely $\mathbf{d}_k \sim \mathcal{N}(\mu_{\mathbf{d}}, \Sigma_{\mathbf{d}})$. However, this can be also some lower-dimensional noise if sensor data is only partially available. The likelihood function can thus be derived from (19) as $f(\mathbf{z}_k|\mathbf{x}_k) = f_{\mathbf{d}_k}(\mathbf{z}_k - h_{\mathcal{M}}(\mathbf{x}_k))$. Moreover, in order to incorporate IMU measurements at the same time, it's possible to augment $\mathbf{z}_k$ into $[\mathbf{d}_k, \mathbf{q}_k]^T$, with $\mathbf{q}_k \in \mathbb{S}^1$ denoting quaternions representing the orientation. We can then assume another non-additive orientation noise on $\mathbb{S}^1$ that is Bingham-distributed and further assume the two noise distributions to be independent such that

$$f(\mathbf{z}_k|\mathbf{x}_k) = f_{\mathbf{d}}(\mathbf{d}_k - h_{\mathcal{M}}(\mathbf{x}_k)) \cdot f_{\mathbf{q}}(\mathbf{x}_{k,r}^{-1} \otimes \mathbf{q}_k) \,.$$

B. Identity Measurement Model

For comparison, we also apply the original filter with the identity measurement model and perform update steps with dual quaternions converted from the same IMU and ultrasonic measurements as in the non-identity case. The identity measurement model is hereby assumed as

$$\mathbf{z}_k = \mathbf{x}_k \otimes \mathbf{v}_k .$$

In order to give the measurement the same uncertainty level as in the non-identity case, we use Monte Carlo sampling of both IMU and ultrasonic measurements and further approximate them to fit a noise distribution in (11). We perform 100 Monte Carlo runs for the simulation with simultaneous measurement of both ultrasonic and IMU. The same system inputs chain is used during each simulation runs. The results are depicted in Fig. 4 showing the pose tracking accuracy. In [15], the originally proposed filter gives better performance than the UKF. Our proposed progressive approach significantly outperforms the originally proposed filter for both orientation and position estimation. This is mainly due to the fact that different sensor data follow their own individual noise distribution. For the identity measurement model, the sensor noise distribution after conversion to the domain of the system state is typically different from the assumed distribution.

VI. Conclusion And Outlook

In this paper, we presented a nonlinear progressive filtering algorithm based on a distribution modeling for uncertain dual quaternions to estimate planar rigid body motion. This approach gives a better consideration of the underlying nonlinear structure of rigid body motion by using dual quaternions. Unlike the former filtering algorithm, where the measurement model is assumed to be identity, our approach can leverage any instantly available measurement data directly with arbitrarily given likelihood function such that the measurement is not necessarily on the same domain of the system state. During the prediction step, our approach employs a deterministic sampling scheme to have better efficiency and give repeatable results compared to random sampling. For performing Bayesian inference with an arbitrary likelihood function during the update step, we use an approach that progressively fuses likelihoods with the prior. We also presented a simulation, which is close to a real-world scenario with measurements from multiple different sensors. By directly fusing measurements with the proper likelihood function, our filter outperforms the original filter, which is designed only for the identity measurement model.

There is still much potential that can be exploited based on our approach. First, we can extend the current planar case to three-dimensional motion. This could be done by extending the current distribution to model uncertain dual quaternions on $\mathbb{S}^3 \times \mathbb{R}^4$ for spatial transformation. For that, a similar progressive update approach can be used for better performance in real-world scenarios. Second, although the distribution used for modeling the noise terms is antipodally symmetric, it still restricts the noise to be uni-modal. This could be solved, e.g., by using a Dirac mixture distribution on the same manifold.

Acknowledgment

This work is supported by the German Research Foundation (DFG) under grant HA 3789/16-1 "*Recursive Estimation of Rigid Body Motions*".

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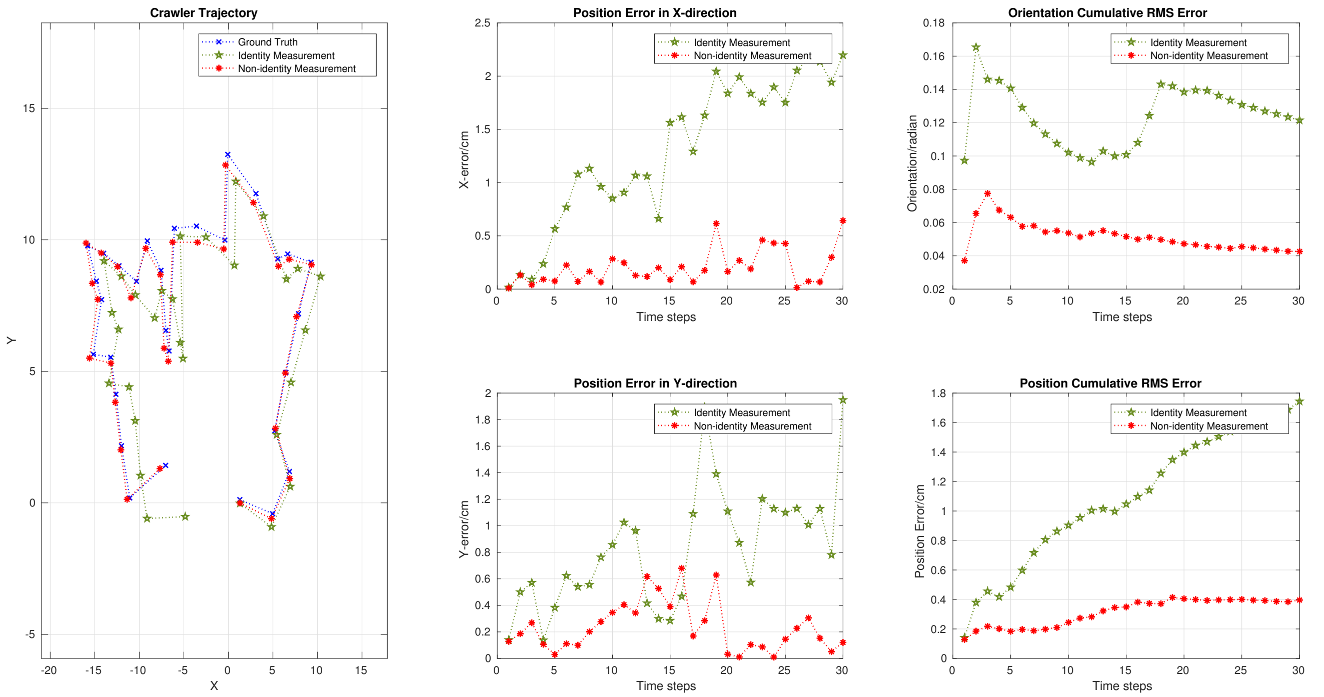

Figure 4: Comparison between the proposed progressive approach (depicted in red) with the original approach (depicted in green) using the identity measurement model from [15]. The evaluation is based on the average of 100 Monte Carlo runs using the same system inputs chain $\{\mathbf{u}_k\}_{k=1,\cdots,30}$. Our approach, which uses a non-identity measurement model (red), outperforms the old filter (green) for both position and orientation estimation. Here we give the temporally cumulative RMS error for position and orientation estimation and keep the raw estimation on x and y to show how their noise evolves over the time steps. Estimated trajectories and ground truth (blue) are visualized on the left.

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