Abstract—We present a novel geometry-driven scheme for generating equally weighted deterministic samples of Bingham distributions in arbitrary dimensions. Unlike existing approaches, our method provides flexibility in the sampling size with samples satisfying requirements of the unscented transform while approximating higher-order moments of the Bingham distribution. This is done by first using Dirac mixture approximation as a sampling scheme on the tangent plane at the mode with respect to the Bingham density via gnomonic projection. Subsequently, the tangent sigma points are retracted backwards to the hypersphere, after which an on-manifold moment correction is performed via Riemannian optimization. The proposed approach is further applied to quaternion Bingham filtering for recursive orientation estimations. Evaluation results show that the geometry-adaptive sampling scheme gives better tracking accuracy and robustness for nonlinear orientation estimations.

I. INTRODUCTION

Recursive orientation estimation plays a crucial role in various applications such as robotic pose estimation [1]–[3], localization and mapping [4] as well as multilateration [5], etc. However, due to the inherently periodic and nonlinear structure of the special orthogonal group SO(3), robust and accurate orientation estimations are nontrivial. Moreover, various options for parameterizing orientations exist. The Euler angles, for instance, enable minimal representations. However, they suffer from ambiguity caused by gimbal lock. The well-known 3 × 3 rotation matrices eliminate the ambiguity via over-parameterization, but introduce a high degree of redundancy (9 elements used for representing 3 DoF), potentially causing numerical instabilities. In contrast, unit quaternions parameterize the spatial orientation without ambiguity and with only one degree of redundancy. Thus, they have been employed for pose representation in various robotic tracking tasks [6], [7].

As unit quaternions are located on the hypersphere S^3, conventional quaternion filters for orientation estimations rely on local linearizations of the nonlinear manifold (e.g., via Lie algebra) [7]. However, this is based on the assumption of local perturbations and might be error-prone for fast rotations under large uncertainty and nonlinearity. By utilizing distributions from directional statistics [8], recent efforts were dedicated to model the uncertainty of unit quaternions directly on the hypersphere without local linearization. For instance, the Bingham distribution on S^3 inherently enables the on-manifold modeling and filtering of unit quaternions for orientation estimations [9]–[11].

However, filtering approaches using the Bingham distribution usually rely on sampling-approximation schemes, in which the approximation is based on the second-order moment matching [9], [12]. Compared to random sampling schemes, deterministic approaches guarantee reproducible results and are in general more efficient [13]. Taking the idea of the unscented transform (UT) [14], the conventional unscented quaternion filter uses sigma points that approximate the Bingham distribution up to the second order. However, the fixed and limited sample size might pose risks for performing filtering under large uncertainty and nonlinearity due to the sample degeneration issue [12]. Therefore, we need more samples drawn adaptively according to the information geometry of the underlying density while satisfying requirements of the UT for moment matching. Moreover, samples with uniform weights are preferred as they can equally contribute during the nonlinear filtering.

In order to flexibly generate a given number of samples in the sense of information geometry [15] for nonlinear filtering, efforts were first devoted to Gaussian distributions in the Euclidean space through Dirac mixture approximation (DMA) [16]–[18]. The basic methodology here is to minimize a certain distance measure between the Dirac mixture density described by the samples and the underlying density under the constraint of second-order moment. The sampling scheme is then formulated as a constrained optimization problem and solved by using standard optimizers (e.g., quasi-Newton method). In [18], samples were drawn in the principal axes for DMA by minimizing the Cramér–von Mises distance. In [19], a modified Cramér–von Mises distance adapted to the localized cumulative distribution (LCD) was proposed for the DMA of multivariate Gaussian distributions. Here, the deterministic samples are not restricted to the principal axes but can be drawn in the entire domain.

1In the context of unscented orientation filtering, 3 and 5 samples are drawn on S^1 (planar case) and S^3 (spatial case), respectively.
Driven by hyperspherical geometry, the first geometry-adaptive sampling scheme for the Bingham distribution was proposed in [20]. Though improved accuracy and robustness have been shown for nonlinear orientation filtering, the samples are still drawn in the principal directions. In this paper, we propose a novel hyperspherical sampling scheme to better approximate higher-order moments of the Bingham distribution. Here, we apply the LCD-based Dirac mixture approximation as a sampling scheme on the tangent plane at the mode with respect to the underlying Bingham density via the gnomonic projection. The tangent sigma points are then retracted to the manifold followed by a moment correction of the second order to satisfy requirements of the UT. The moment correction is formulated as a manifold optimization problem and solved by the Riemannian steepest descent algorithm [21]. More specifically, we summarize our main contributions as follows:

- A generic deterministic sampling scheme is proposed for Bingham distributions in arbitrary dimensions with equally weighted samples of flexible size.
- The samples drawn on the hypersphere (not only in principal directions) satisfy requirements of the unscented transform and give approximation to higher-order moments of the Bingham distribution.
- By using the proposed sampling scheme, better accuracy and robustness are shown in nonlinear Bingham filtering for orientation estimations.

The remainder of the paper is structured as follows. In Sec. II, preliminaries about quaternion-based orientation representation and the Bingham distribution as well as the LCD-based DMA are introduced. In Sec. III, we propose the novel deterministic sampling approach for the Bingham distribution based on Riemannian geometry. The proposed sampling scheme is further evaluated for nonlinear Bingham filtering in Sec. IV. The work is finally concluded in Sec. V.

II. PRELIMINARIES

A. Unit Quaternion Parameterization for Spatial Orientations

By convention, orientation states belonging to the special orthogonal group $SO(3)$ can be represented by the unit quaternion as follows:

$$\mathbf{x} = \left[ \cos(\theta/2), \mathbf{u}^\top \sin(\theta/2) \right]^\top,$$

with the unit vector $\mathbf{u}$ being the rotation axis and $\theta$ the rotation angle. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be rotated around axis $\mathbf{u}$ of degree $\theta$ by (1) according to

$$\mathbf{v}' = \mathbf{x} \otimes \mathbf{v} \otimes \mathbf{x}^*, \tag{2}$$

with $\otimes$ denoting the Hamilton product and $\mathbf{x}^*$ the conjugate of $\mathbf{x}$ [5], [22], namely $\mathbf{x}^* = \text{diag}(1, -1, -1, -1) \mathbf{x}$. Unit quaternions are thus naturally located on the hypersphere $S^3 \subset \mathbb{R}^4$ because of the unit norm constraint. Furthermore, two antipodal quaternions on the hypersphere, i.e., $\mathbf{x}$ and $-\mathbf{x}$, denote the same rotation as shown in (2). Therefore, a density $f$ for stochastically modeling uncertain unit quaternions should be antipodally symmetric on the hypersphere, such that $f(\mathbf{x}) = f(-\mathbf{x})$.

B. Bingham Distribution

Intuitively, the Bingham distribution on $S^n$ can be derived by constraining a zero-mean Gaussian distribution of $\mathbb{R}^{n+1}$ on the hypersphere followed by a re-normalization. Traditionally, it is defined as follows

$$f_B(\mathbf{x}; \mathbf{Z}, \mathbf{M}) = \frac{1}{N(\mathbf{Z})} \exp \left( \mathbf{x}^\top \mathbf{Z} \mathbf{M} \mathbf{Z}^\top \mathbf{x} \right), \quad \mathbf{x} \in S^n, \tag{3}$$

with the diagonal matrix $\mathbf{Z}$ determining the concentration as well as the normalization constant $N(\mathbf{Z})$, and the real orthogonal matrix $\mathbf{M}$ controlling the orientation of the distribution on the hypersphere $S^n \subset \mathbb{R}^{n+1}$. The parameters $\mathbf{Z}$ and $\mathbf{M}$ can be derived via eigendecomposition of a negative semidefinite matrix $C_B \in \mathbb{R}^{(n+1) \times (n+1)}$. Afterwards, elements of $\mathbf{Z}$ are by convention adjusted in ascending order, namely $\mathbf{Z} = \text{diag}(z_1, \ldots, z_n, 0)$, with $z_1 < \ldots < z_n < 0$, followed by a reordering of the column vectors in $\mathbf{M}$ accordingly. The mode of the Bingham distribution is thus denoted by the last column of $\mathbf{M}$ as it corresponds to the largest eigenvalue in the space $\mathbb{R}^{n+1}$. Furthermore, the Bingham distribution is inherently antipodally symmetric on $S^n$ since $f_B(\mathbf{x}) = f_B(-\mathbf{x})$. For orientation estimations using unit quaternions, the Bingham distribution on $S^3$ and $\mathbb{S}^1$ can therefore be employed for stochastic modeling uncertain spatial and planar orientations, respectively.

C. Dirac Mixture Approximation for Multivariate Gaussian Distribution Based on the Localized Cumulative Distribution

In [19], [23], multivariate Gaussian distributions were approximated with the following Dirac mixture

$$f(x) = \sum_{j=1}^{m} w_i \cdot \delta(x - \mathbf{x}_j), \tag{4}$$

with $\delta(\cdot)$ being the Dirac delta function. Here, $\mathbf{x}_j$ denotes the location of each Dirac component and $w_i \in [0, 1]$ the weighting factor of each sample (for uniformly weighted samples $w_i = 1/m$). For an arbitrarily given probability density function $g : \mathbb{R}^n \to \mathbb{R}_+$, its localized cumulative distribution (LCD) is defined as follows

$$\mathcal{F}(\mathbf{r}, \mathbf{b}) = \int_{\mathbb{R}^n} g(\mathbf{x}) \cdot \kappa(\mathbf{x} - \mathbf{r}, \mathbf{b}) d\mathbf{x}, \tag{5}$$

with $\kappa(\mathbf{x} - \mathbf{r}, \mathbf{b})$ denoting a suitable kernel (symmetric and integrable) that locally measures the density at $\mathbf{m}$ in a range of $\mathbf{b}$. In [19], [24], axis-wise separable kernels were used, i.e.,

$$\kappa(\mathbf{x} - \mathbf{r}, \mathbf{b}) = \prod_{i=1}^{n} \kappa(x_i - r_i, b_i),$$

with each kernel suggested to be of Gaussian type with isotropic dispersion (or can be written as a scalar value), thus

$$\kappa(x_i - r_i, b) = \prod_{i=1}^{n} \exp \left\{ - \frac{1}{2} \frac{(x_i - r_i)^2}{b^2} \right\}.$$
Based on the LCD defined in (5), a modified Cramér–von Mises distance is proposed to measure the distance between the Dirac mixture \( f(x) \) given in (4) and the underlying Gaussian distribution \( \hat{f}(x) \). This distance measure is given in the following form

\[
D = \int_{\mathbb{R}^+} w(b) \int_{\mathbb{R}^n} (\hat{F}(r, b) - F(r, b)) \, dr \, db, \tag{6}
\]

with \( \hat{F}(m, b) \) and \( F(m, b) \) being the LCD of the underlying Gaussian distribution and the Dirac mixture, respectively. Here, \( w(b) \) is a weighting function determined by the range factor for the kernel and is given by

\[
w(b) = \begin{cases} b^{1-n}, & b \in [0, b_{\text{max}}] \\ 0, & \text{elsewhere} \end{cases}.
\]

As proposed in [19], the Dirac mixture approximation to a multivariate Gaussian distribution can thus be formulated as a constrained optimization problem, where the distance measure in (6) is minimized and the second-order moment constraint should be satisfied.

The aforementioned LCD-based DMA for multivariate Gaussian distribution can be efficiently done with closed-form gradient and error term in either an online or offline manner [19]. In this way, deterministic samples of arbitrary size that satisfy the second-order moment constraint and approximate the higher-order moments can be generated. For nonlinear recursive estimation in the Euclidean space, the LCD-based adaptive sampling scheme has facilitated the development of trackers with better accuracy and higher robustness [25].

III. RIEMANNIAN SPHERICAL SAMPLING (RSS)

For the DMA-based sampling approach, a proper kernel function should be defined on the domain of the distribution, based on which a certain error metric between the Dirac mixture and the underlying density can be formulated and minimized [17], [19]. However, this scheme cannot be trivially applied on manifolds. The kernel function used to evaluate the density in the Euclidean space is normally an isotropic Gaussian. On the hypersphere, the Gaussian kernel should be modified according to the geometric structure of the nonlinear domain. A Bingham or von Mises–Fisher kernel can be employed for this purpose. However, the subsequent error metric formulation normally refers to an integral on the nonlinear domain. For Bingham distributions, this boils down to an integral over the hypersphere, which leads to a similar or even larger computational cost than calculating the normalization constant [26]. This is, obviously, against the spirit of pursuing good efficiency via deterministic sampling for recursive estimations.

In order to alleviate the computational burden, we exploit hyperspherical geometry to enable the LCD-based DMA on the tangent plane at the Bingham mode. As pointed out in [20], projections or the logarithm map can be employed to derive the on-tangent-plane density from the underlying hypersphere. We hereby apply the gnomonic projection [27], as it results in unbounded domain of the projected Bingham density on the tangent plane. This is coherent to the definition of the LCD and paves the way to retract tangent sigma points back to the hypersphere.

A. Gnomonic Projection

As the parameter matrix \( M \) of the Bingham distribution in (3) is orthogonal, the first \( n \) columns of matrix \( M \) provide an orthonormal basis of the tangent plane at the mode of the Bingham distribution, namely

\[
T_M \mathbb{S}^n = \text{span}\{m_1, \ldots, m_n\},
\]

with \( M = [m_1, \ldots, m_n] \in \mathbb{R}^{(n+1) \times (n+1)} \) and \( m \) being the mode. For better readability, we denote the tangent local basis as \( E = [m_1, \ldots, m_n] \in \mathbb{R}^{(n+1) \times n} \) such that \( M = [E, m] \). For any \( x \in T_M \mathbb{S}^n \subset \mathbb{R}^n \), its coordinate expressed in the local basis \( E \) is thus

\[
x_i = E^\top x_i.
\]

For any \( x \in \mathbb{S}^n \), it can be mapped from the hypersphere to the point \( x_i \) on the tangent plane via the gnomonic projection
We therefore substitute \( x \). We can reformulate the Bingham density under gnomonic projection with respect to the local coordinate on \( T_mS^n \) into the following form
\[
\mathcal{P}_m : S^n \rightarrow T_mS^n \quad [20]. \]
Its inverse operation is essentially a retraction that project \( x \) from the tangent plane back to the hypersphere, namely \( \mathcal{R}_m : T_mS^n \rightarrow S^n \), according to
\[
\mathcal{R}_m(x_i) = \frac{x_i + \mathbf{m}}{\sqrt{1 + ||x_i||^2}}. \tag{8}
\]

\[ \mathbf{E} = \begin{cases} 2.7 \times 10^{-13} \quad m = 10 \\ 4.0 \times 10^{-12} \quad m = 30 \\ 1.4 \times 10^{-12} \quad m = 50 \end{cases} \]
(a) Bingham distribution on \( S^1 \) (left) with \( \mathbf{C}_B = \text{diag}(1,5) \) and on \( S^2 \) (right) with \( \mathbf{C}_B = \text{diag}(1,5,5) \).

\[ \mathbf{E} = \begin{cases} 8.6 \times 10^{-15} \quad m = 10 \\ 1.7 \times 10^{-11} \quad m = 30 \\ 2.4 \times 10^{-11} \quad m = 50 \end{cases} \]
(b) Bingham distribution on \( S^1 \) (left) with \( \mathbf{C}_B = \text{diag}(1,10) \) and on \( S^2 \) (right) with \( \mathbf{C}_B = \text{diag}(1,5,10) \).

\[ \mathbf{E} = \begin{cases} 5.7 \times 10^{-13} \quad m = 10 \\ 5.8 \times 10^{-12} \quad m = 30 \\ 5.7 \times 10^{-12} \quad m = 50 \end{cases} \]
(c) Bingham distribution on \( S^1 \) (left) with \( \mathbf{C}_B = \text{diag}(1,30) \) and on \( S^2 \) (right) with \( \mathbf{C}_B = \text{diag}(1,5,40) \).

\[ \mathbf{E} = \begin{cases} 4.2 \times 10^{-13} \quad m = 50 \\ 1.0 \times 10^{-11} \quad m = 200 \\ 6.0 \times 10^{-11} \quad m = 500 \end{cases} \]

\[ \mathbf{E} = \begin{cases} 4.2 \times 10^{-13} \quad m = 50 \\ 1.0 \times 10^{-11} \quad m = 200 \\ 6.0 \times 10^{-11} \quad m = 500 \end{cases} \]

\[ \mathbf{E} = \begin{cases} 4.2 \times 10^{-13} \quad m = 50 \\ 1.0 \times 10^{-11} \quad m = 200 \\ 6.0 \times 10^{-11} \quad m = 500 \end{cases} \]

**B. On-Tangent-Plane Approximation to On-Manifold Density**

In order to perform LCD-based DMA on the tangent plane at the mode, we use the gnomonic projection to map the density from the hypersphere to the tangent space at the mode. We therefore substitute \( x \) in (3) by its projected point via the gnomonic retraction given in (8), such that the projected Bingham can be derived as follows
\[
f(x_i) = f_B(\mathcal{R}_m(x_i)) \\
= \frac{1}{N(Z)} \exp \left\{ \frac{(x_i + \mathbf{m})^T \mathbf{M} \mathbf{Z}^T (x_i + \mathbf{m})}{1 + ||x_i||^2} \right\} \\
= \frac{1}{N(Z)} \exp \left\{ \frac{x_i^T \mathbf{M} \mathbf{Z}^T x_i}{1 + ||x_i||^2} \right\}. \tag{9}
\]

As the last column of \( \mathbf{M} \) is orthogonal to the tangent plane \( T_mS^n \) and the last diagonal element of \( \mathbf{Z} \) is set to be zero, we can reformulate the Bingham density under gnomonic projection with respect to the local coordinate on \( T_mS^n \) into the following form
\[
f(\tilde{x}_i) = \frac{1}{\beta} \exp \left\{ \frac{\tilde{x}_i^T \mathbf{Z}_i \tilde{x}_i}{1 + ||\tilde{x}_i||^2} \right\}, \quad \text{with} \quad \tilde{x}_i = \mathbf{E}^T x_i \in \mathbb{R}^n, \quad \mathbf{Z}_i = \text{diag}(z_1, \ldots, z_n) \in \mathbb{R}^{n \times n}.
\]

Here, \( \beta = N(Z) \) denotes the Bingham normalization constant.

The function in (9) provides the density value at \( \tilde{x}_i \) according to the underlying Bingham on the hypersphere via the gnomonic projection. However, to perform deterministic sampling using the LCD-based DMA approach introduced in Sec. II-C, the density should be converted to or approximated by a Gaussian distribution. Essentially, the denominator in (9) denotes the squared cosine of the angle induced by the projection line, namely \( \cos(\alpha)^2 = 1/(1 + ||x_i||^2) \), with \( \alpha = x^T \mathbf{m} \). By applying Taylor expansion to \( \alpha \), the density can be approximated by \( f(\tilde{x}_i) \approx \frac{1}{\beta} \exp \{ \tilde{x}_i^T \mathbf{Z}_i \tilde{x}_i \} \). This density can be further converted into an isotropic zero-mean Gaussian by imposing the normalization constant \( \beta \) to be equal to the covariance
\[ \mathbf{\Omega} = \frac{\beta^2/n}{2\pi} \mathbf{I}, \tag{10} \]

such that \( \beta = \sqrt{\det(2\pi \mathbf{\Omega})} \). We can therefore perform the LCD-based sampling approach introduced in Sec. II-C to...
We therefore propose to formulate the moment correction as
\[
\text{arg min}_{\sigma_j} \| \sigma_j - \text{cov} \|_F^2
\]
where the distance between the covariance of the sigma points
is explicitly considered in each iteration. As Riemannian optimizers are geometry-aware, a better convergence speed
and robustness can be expected [21].

Moreover, to let the samples after moment correction
maintain the higher-order shape information given by the LCD-based DMA as much as possible, the location change of each
sample due to the correction should also be restricted. We
thus recommend to use the Riemannian steepest descent (RSD) algorithm with raw retracted sigma points as the initialization.

Another solution to overcome convergence to a global minimum
of large distortion is to add a penalty term to (13), such that
\[
E(\{\sigma_j\}_{j=1}^m) = \frac{1}{m} \| \sigma_j \sigma_j^\top - \text{cov} \|_F^2 + \lambda \frac{m}{2} \sum_{j=1}^m \sigma_j \sigma_j^\top m
\]
The second term in the equation above denotes a scaled (by \(\lambda\))
average location change on the hypersphere compared with the initialization. For this penalty term, the distance metric from
von Mises-Fisher distribution is used\(^3\).

\(^3\)Obviously, other metrics (e.g., Euclidean distance) can be also employed.

**Algorithm 1 Riemannian Spherical Sampling**

```
procedure RSS \((f_B, m)\):
1: \(n \leftarrow \text{getHypersphereDimension} \((f_B)\); \)
2: \(\beta \leftarrow \text{getNormalizationConstant} \((f_B)\); \)
3: \(E \leftarrow \text{getLocalCoord} \((f_B)\); \)
4: \(\Omega \leftarrow \frac{\beta^{2/n}}{2\pi} I; \) // see (10)
5: \(\{u_j\}_{j=1}^m \leftarrow \text{LCD-DMA} \((N(0, \Omega), m)\); \)
6: for \(j = 1\) to \(m\) do
7: \(\sigma_j \leftarrow -\sqrt{\frac{\beta^{2/n}}{2\pi}} Z, u_j; \) // see (11)
8: \(\sigma_j = R_m(E \sigma_j); \) // see (12)
9: end for
10: \(\{\sigma_j\}_{j=1}^m \leftarrow \text{momentCorrection} \((\{\sigma_j\}_{j=1}^m\); \) // see (13)
11: return \(\{\sigma_j\}_{j=1}^m\)
end procedure
```
For varying dimensionality and parameter configurations, Fig. 2 shows the sampling results given by the proposed approach on $S^3$ with various parameter matrices $C_B$ introduced in Sec. II-B and sample size $m$ as follows.

- $S^1$: parameter $C_B = -\text{diag}(1, a)$ with $a = 5, 10, 30$ and sample size $m = 10, 30, 50$.  
- $S^2$: parameter $C_B = -\text{diag}(1, 5, a)$ with $a = 5, 10, 30$ and sample size $m = 50, 200, 500$.  
- $S^3$: parameter $C_B = -\text{diag}(1, 5, 5, 5)$ and sample size $m = 50, 200, 500$.

Fig. 2 shows the sampling results given by the proposed approach on $S^3$ and $S^2$ and results for $S^3$ are shown in Fig. 3. For varying dimensionality and parameter configurations, the proposed sampling approach can generate deterministic samples maintaining the second-order moment (denoted by $E$) while being adaptive to the higher-order moments of the Bingham distribution.

### Example III.1
We apply the RSS for Bingham distributions of different dimensions with various parameter matrices $C_B$ introduced in Sec. II-B and sample size $m$ as follows.

- $S^1$: parameter $C_B = -\text{diag}(1, a)$ with $a = 5, 10, 30$ and sample size $m = 10, 30, 50$.  
- $S^2$: parameter $C_B = -\text{diag}(1, 5, a)$ with $a = 5, 10, 30$ and sample size $m = 50, 200, 500$.  
- $S^3$: parameter $C_B = -\text{diag}(1, 5, 5, 5)$ and sample size $m = 50, 200, 500$.

The residual value $E$ after correction is defined in (13).

which is the orientation $z_k$ rotated from the initialization $v_0$. We assume the noise $v_k$ to be additive and follow a zero-mean Gaussian distribution, i.e., $v_k \sim \mathcal{N}(0, \Omega)$. As proposed in [5], [12], for quaternion Bingham filtering, the prediction step can be performed via the composition of Bingham distributions. For the non-identity measurement model in (15), the posterior is normally approximated from the prior samples that are reweighted according to the likelihoods

$$f(z_k | x_k) = f_{\mu_k} (z_k - x_k \otimes n_k \otimes x_k^*) .$$

It can therefore be risky to rely on a small number of samples for performing the update step. For instance, the UT-based approach from [12] can only generate seven samples, which can easily degenerate after the reweighting. A typical method to overcome the sample degeneration issue is to apply the progressive update step [5], [33], which gradually fuse the measurement with the prior. However, the existing progressive Bingham filter [5] still relies on the UT-based deterministic sampling approach, leading to an ineffective progression step as the non-mode samples can also easily degenerate. For the following evaluation cases, we apply the proposed Riemannian spherical sampling (RSS) approach in a simple sampling-approximation Bingham filtering scheme (without progressive update) for spatial orientation estimation. For each of the cases, we perform 100 Monte Carlo runs of 50 time steps.

First, we evaluate the proposed RSS-based Bingham filtering in comparison with the unscented Bingham filter (UBF) from [12] and the progressive UBF from [5]. Here, the system noise $w$ is Bingham-distributed with the parameter $C_B = -\text{diag}(1, 10, 10, 10)$. The additive measurement noise follows zero-mean Gaussian distributions with different uncertainty levels of $\Omega_w = a \times \text{diag}(1, 10, 10, 10)$, where $a = 10^{-1}, 10^{-2}$ and $10^{-4}$.

We use $m = 100$ deterministic samples for the RSS-based filter. As shown in Fig. 7, for all levels of measurement uncertainties the proposed approach shows better tracking accuracy than the other two existing approaches. Specifically, for low measurement noise, where samples are prone to degenerate due to the peaky likelihood function, the UT-based
Fig. 7: Evaluation of the proposed deterministic sampling approach for quaternion-based orientation estimation under different measurement noise levels (shown by box plot of MATLAB). We compare the Bingham filter (BF) based on the RSS (100 deterministic samples, shown as RSS-100) with the unscented BF [12] (7 deterministic samples satisfying requirements of the unscented transform (UT), shown as UT-7) and the UT-based progressive BF [5] (shown as UT-Prog-7). The proposed deterministic sampling approach enables the most robust and accurate tracking performance using the non-progressive filtering scheme with only a sampling-approximation-based update step.

Fig. 8: Evaluation of Bingham filtering for orientation estimation using the proposed deterministic sampling approach with different sample sizes (shown by box plot of MATLAB). The proposed RSS shows improved tracking accuracy and higher runtime with increasing sample size. Compared with the Bingham particle filter [32] (using 2000 random samples, shown as Rnd-2000), the proposed RSS enables good combination of tracking accuracy and efficiency.

approach almost totally failed with several outliers showing large error. However, the Bingham filter based on the proposed sampling approach works robustly and most accurately. Fig. 6 further shows an example of the visualized tracking result given by Bingham filters based on different sampling approaches.

Second, we evaluate the proposed RSS-based Bingham filtering with different sample sizes of $m = 20, 50, 100$ under medium measurement noise. Additionally, we employ a naively implemented particle filter based on 2000 random samples [32] as a reference. Fig. 8 shows that the RSS-based Bingham filter improves tracking accuracy with larger size of deterministic samples. Due to the online optimization-based sampling scheme, the RSS-based Bingham filter shows worse efficiency with larger sample size. Compared to the Bingham particle filter with 2000 samples, the RSS-based Bingham filter with 100 deterministic samples gives accurate enough results with much less runtime.

V. CONCLUSION

In this paper, we introduced a novel deterministic sampling approach that can be applied to Bingham distributions in arbitrary dimensions for any given sample size. Unlike existing sampling schemes, the proposed approach satisfies requirements of the unscented transform (preserve the first two moments) and approximates the higher-order moments of the Bingham distribution with samples drawn on the hypersphere. Therefore, our novel approach enables improved nonlinear Bingham filtering for quaternion-based orientation estimations.

However, there is still much potential to exploit for the presented work. The proposed sampling scheme gives DMA to an on-tangent-plane Gaussian distribution, which is an approximation of the true projected Bingham density. It is thus valuable to further investigate the difference between drawing samples from the approximated density and the true one. Moreover, by applying the proposed sampling approach to the Bingham-based pose filtering frameworks [4], [6], [34]–[36], a better performance regarding the tracking accuracy, efficiency, and robustness can be expected.

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