Approximate First-Passage Time Distributions for Gaussian Motion and Transportation Models

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Abstract—We aim to approximate the distribution of the first-passage time of a particle moving according to a Gaussian process with increasing trend, i.e., the distribution of the first time a particle described, e.g., by a state-space model such as a constant-velocity or constant-acceleration model, arrives at a fixed location. Since the known approaches from the literature either consider processes from different families or lead to highly complex approximations, we seek a fast-to-compute method for the problem. Motivated by an engineering particle transport task for which we can assume that once a particle has arrived at this location it cannot move back, we derive an analytic approximation for the first-passage time probabilities and calculate its inverse cumulative distribution function analytically and the moments numerically. Furthermore, we propose a Gaussian approximation based on a linearization approach. The strengths and limitations of our methods are discussed and by comparison with Monte Carlo simulations, we show that in particular, the first one satisfies the requirements of engineering problems in terms of accuracy and computation time.

Index Terms—First passage, Gaussian process, probabilistic motion model, absorbing boundary, kinematic model.

I. INTRODUCTION

The problem of finding the distribution when a particle moving according to a stochastic process hits a fixed, moving, or curved boundary for the first time is called a first-passage or first-hitting time problem [1], [2]. Such problems are typically encountered in the study of diffusion processes in physics and biology but also arise, for example, in psychology, economics, signal processing, or engineering (see e.g. [3]–[5] and references therein).

In our example, we are originally motivated by the technical application of particle sorting [6], in which particles are transported by a conveyor belt while being tracked using a Kalman filter. We aim to predict the first-passage time w.r.t. an array of nozzles that is located close behind the end of the belt and ejects particles that are not of the desired class with jets of compressed air (see Fig. 2). As the filter provides us with estimated particle positions, velocities, and possibly accelerations at the end of the belt and we assume a constant velocity (CV) or constant acceleration (CA) motion model in transport direction, mathematically, this can be seen as a first-passage time problem w.r.t. a fixed boundary under stochastic initial conditions for a non-stationary, non-centered stochastic process.

Similar scenarios, where the motion of a particle or an agent is modeled by a Gaussian stochastic process with an increasing trend, i.e., an increasing mean function, are prevalent in science and engineering. One reason is that kinematic motion models perturbed by Gaussian random noise, such as CV or CA models, are still a core part of almost all applications involving state estimation or prediction. For example, such models are usually found in the field of target tracking and navigation [7], [8], autonomous driving and pedestrian motion prediction [9], or robot navigation and odometry [10]. First-passage time problems may arise there naturally, e.g., when trying to predict the time of arrival of an agent or the remaining time until a collision, where additional uncertainty information can be crucial.

Although first-passage time problems have been studied since the early 1900s, they still remain a class of challenging problems and analytic solutions are only known for some special combinations of mainly basic processes and boundaries,
e.g., Brownian motion with a fixed boundary. For general linear Gaussian state-space models (LGSSM), such as CV and CA models, no analytic solutions are known. Furthermore, the known approximation schemes for the first-passage time distribution (FPTD) and its moments are mostly designed for diffusion processes and only often available for special kinds of processes (e.g., one-dimensional, stationary, centered, or Markov). Even with the known approaches to related problems, it is our understanding that for most engineering applications, it is hard to obtain moment or quantile information within a reasonable computational time, so they do not provide a viable solution from an engineering perspective, either. Moreover, they do not take into account the characteristics typical for many motion models in engineering, such as 1) the increasing trend in the mean function and 2) the (compared with diffusion processes) generally weakly-stochastic short-time behavior. Both characteristics may allow for further approximations to achieve a good balance between approximation accuracy and computation time.

We propose two methods to tackle the problem in our engineering example: The first uses a linearization approach and assumes a Gaussian density for the FPTD whose mean and variance are calculated from the linearization. The second one exploits that particles are very unlikely to move back after passing the nozzle array, since otherwise efficient sorting would not be possible. Under this assumption, we can work with the formulas for first-passage time problems with absorbing boundaries and calculate a simple, analytic approximation for the first-passage time probabilities (see Fig. 1) without any further linearizations or assumptions on the type of distribution.

Contribution: First, we propose a simple Gaussian approximation scheme to the FPTD of LGSSMs with an increasing trend called Gauss–Taylor approximation. Second, we clarify the mathematical requirements that the process-boundary pair must satisfy to ensure the assumption of non-returning particles. Using this assumption, we then propose a simple analytic approximation to the FPTD called No-return approximation that is applicable to Gaussian processes with an increasing trend. Third, we derive the analytic solution for the corresponding first-passage time problem with absorbing boundaries and continuous-time Gaussian processes with an increasing trend, i.e., with a continuously increasing mean function at time.

In particular, we consider time-invariant continuous-time LGSSMs without input, i.e., Markov dynamic systems that are described by the Langevin equation
\[
\dot{x}(t) = \hat{A}x(t) + D \eta(t).
\]
Here, \(\eta(t)\) is a continuous, stationary, zero-mean white noise with power spectral density \(V\), \(\hat{A}\) is the system matrix, \(D\) is the noise gain, and \(x(t)\) is the state of the LGSSM.

II. Problem Statement

The first-passage time is defined as the first time a (continuous) random process \(x(t)\) reaches a (potentially time-dependent) value \(a(t)\), i.e., \(T_a = \inf\{t > t_0 : x(t) = a(t)\}\), when starting at a time \(t_0\) at \(x(t_0) < a(t_0)\) almost surely.\(^2\) Our goal is to calculate the probability \(\mathbb{P}(T_a < t)\), i.e., the CDF of \(T_a\) or its PDF (the FPTD) \(p_{T_a}(t)\), if they exist. The latter requires that the first passage is an almost sure event, which might not be guaranteed for any arbitrary pairing of processes and boundaries. In this paper, we focus on fixed boundaries \(a(t) = a \in \mathbb{R}, \forall t\) and continuous-time Gaussian processes with an increasing trend, i.e., with a continuously increasing mean function \(\dot{x}(t)\) at time.

Notation: Vectors will be indicated by underlined letters, e.g., \(\underline{x}\), and boldface capital letters, e.g., \(A\), will indicate matrices. We use boldface letters, \(x\), to represent random variables and time \(t\) dependencies of (random) variables, \(x(t)\), will be written in parentheses. As abbreviations, we use \(\hat{x}\) for the expected value of \(x\), i.e., \(\hat{x} = \mathbb{E}\{x\}\) and the upper index \(t_0\) to denote initial values (at time \(t_0\)), e.g., \(x^{t_0} = x(t_0)\).

\(^1\)Our source code is available at https://github.com/KIT-ISAS/Approx_FPTD_for_Motion_Models/tree/Fusion23

\(^2\)The case where \(x(t_0) > a(t_0)\) almost surely is defined analogously.
and autocovariance $\Phi(t-s)\text{Cov}(\mathbf{x}(s))$, $t_0 \leq s \leq t$ with $\Sigma^0$ the initial covariance, i.e., $\Sigma^0 = \text{Cov}(\mathbf{x}(t_0))$ (assuming $\mathbf{x}(t_0)$ to be independent of the noise) [11]. Throughout this paper, we assume w.l.o.g. that (the position) $x(t)$ is the first component of the LGSSM’s state $\mathbf{x}(t)$.

**Example 1 (CV Model).** The continuous-time CV model, also known as the white noise acceleration model, is the LGSSM with state $\mathbf{x}(t) = [x(t) \dot{x}(t)]^\top$, and system matrix and noise gain

$$
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top,
$$

where we have used $x(t)$ and $\dot{x}(t)$ to denote the particle’s position at time $t$ and its first temporal derivative, respectively. The system noise $v(t)$ is therefore scalar-valued with power spectral density $S$. Calculation of $\Phi(\tau)$ yields $[1 \tau \top]$. The mean of the process evolves according to $\dot{x}(t) = \dot{x}^0 + \dot{x}^0 (t - t_0) \dot{x}^0$ and the evolution of the covariance is given by

$$
S \begin{bmatrix} (t-t_0)^3 & (t-t_0)^2 \\ (t-t_0)^2 & t-t_0 \end{bmatrix} + \begin{bmatrix} \Sigma^0_{xx} + 2\Sigma^0_{x\tau} (t-t_0) + \Sigma^0_{\tau\tau} (t-t_0)^2 & \Sigma^0_{xx} + \Sigma^0_{x\tau} (t-t_0) \\ \Sigma^0_{xx} + \Sigma^0_{x\tau} (t-t_0) & \Sigma^0_{xx} + \Sigma^0_{\tau\tau} (t-t_0) \end{bmatrix},
$$

with $\Sigma^0_{xx}$, $\Sigma^0_{x\tau}$, and $\Sigma^0_{\tau\tau}$ denoting the respective entries of $\Sigma^0$.

### III. Background and Related Work

We now briefly summarize the basics of first-passage time problems and the related work for process-boundary pairs that are closest to our problem statement. For some classical results on first-passage time problems and the related work for process-boundary pairs, we refer the reader to the surveys [1] and [2].

A. Background on First-Passage Times

We can find a starting point by writing the event $\{T_a < t\}$ as the union of $\{x(t) > a\}$ and $\{T_a < t, x(t) \leq a\}$, that is, to have a first-passage time $T_a$ less than $t$, a particle must be located either above $a$ at time $t$ or, if it is below $a$, its path must have crossed the boundary at a time $T_a < t$ [12]. As both events are disjoint, it follows

$$
\mathbb{P}(T_a < t) = \mathbb{P}(x(t) > a) + \mathbb{P}(T_a < t, x(t) \leq a).
$$

Note that the first term on the right-hand-side of (3) is easy to compute for Gaussian processes, whereas the second term can only be computed in some special cases.

1) Method of Images: The method of images can be used to find the FPTD for Markov processes that are symmetric around the boundary [5], such as the Brownian motion $b(t)$ with a fixed boundary. When restarting the Brownian motion at the first-passage time $T_a$, it is equally likely that the process ends at time $t$ below or above $a$, thus $\mathbb{P}(T_a < t, b(t) \leq a) = \mathbb{P}(T_a < t, b(t) > a) = \mathbb{P}(b(t) > a)$. This is also known as the reflection principle and follows from the strong Markov property of Brownian motion (see [13], [14] for a proof). Thus,

$$
\mathbb{P}(T_a < t) = 2\mathbb{P}(b(t) > a) = 2 - 2\Phi\left(\frac{a}{\sqrt{t}}\right),
$$

where $\Phi(\cdot)$ is the standard Gaussian CDF and $\sigma^2$ is the diffusion constant of Brownian motion. From (4), we can calculate the FPTD $p_{T_a}(t)$ by taking the derivative w.r.t. $t$, which results in the Lévy distribution. Note that as $t \to \infty$, $\mathbb{P}(T_a < t) \to 1$, which means that the first passage with an arbitrary boundary $a$ is a sure event in case of Brownian motion.

2) Renewal Theory: For Markov processes, an integral equation can be formulated that must be satisfied by the FPTD. Let us suppose that a particular sample path crosses the border at least once between time $t_0$ and $t$, i.e., $x(t_0) \leq a \leq x(t)$. We denote the sample path’s first-passage time by $\theta$. It follows $x(\theta) = a$. Then, due to the Markov property,

$$
p(x(t)|x(t_0)) = \int_{t_0}^{t} p_{T_a}(\theta|x(t_0))p(x(t)|x(\theta))\,d\theta,
$$

if $x(t) \geq a$. This integral equation is often referred to as renewal equation [1]. For some special process-boundary combinations, it is possible to solve (5) analytically (see, e.g., [1] for the Brownian motion and a fixed boundary). Since the integral of the right-hand-side of (5) is a convolution integral, (5) can be solved in Laplacian space. However, the inverse Laplace transform rarely exists, and in general, it is not possible to solve (5) for arbitrary Gauß–Markov processes. Using the Laplace transform of the renewal equation, it is, e.g., possible to find the FPTD for the Wiener process with drift $w(t) = vt + b(t)$, which results in the Inverse Gaussian distribution [15].

3) Absorbing Boundaries: Until now, we considered free particle motion, which means that the presence of the boundary does not affect particle motion. Opposed to that, at an absorbing boundary, a particle immediately stops moving. It thus imposes an additional boundary condition the process PDF has to fulfill (for Markov processes, one can enforce these implications when deriving the PDF via the Fokker–Planck equation). For absorbing boundary conditions and Markov processes, one can again find a renewal integral equation [3]

$$
\int_{-\infty}^{a} p(x(t)|x(t_0))\,dx(t) = 1 - \int_{t_0}^{t} p_{T_a}(\theta|x(t_0))\,d\theta.
$$

This follows from the observation that with an absorbing boundary, the probability of not having passed the boundary by time $t$ is equal to the probability that the first-passage time is greater than $t$.

B. Related Work

Here, we restrict ourselves to the related work in the field of first-passage time problems for the integrated Brownian motion $b(-1)(t) = x_0 + \int_0^t v(\tau)\,d\tau$ with $v(t) = v_0 + b(t)$
and deterministic initial values \( x(0) = x_0 \) and \( v(0) = v_0 \). This is because integrated Brownian motion is closest to LGSSMs since with the state \( \mathbf{z}(t) = \left[ b^{-1}(t) \mathbf{v}(t) \right] ^\top \) one can find a space-state representation known as the Kolmogorov diffusion [16]. In particular, the CV model can be seen as a (more general) version of the integrated Brownian motion with stochastic initial conditions (note that both follow exactly the same Langevin equation and thus also the transition density and the autocovariance match). Furthermore and not restricted to integrated Brownian motion, we outline some approximation methods for estimating FPTDs.

1) Integrated Brownian Motion: Early studies on the integrated Brownian Motion with initial values \( x_0 = 0 \) and \( v_0 > 0 \) are the ones from McKean [17] and Goldman [18], who studied the first return-time to the origin. In 1985, Marshall and Watson [19] succeeded in finding the Laplace transform of the FPTD for general \( x_0, v_0 \), which allows studying the short and long-time behavior of the FPTD. However, until now, no analytic solution is known to the inverse Laplace transform. For a more comprehensive review of the relevant literature in this field, we refer the reader to the survey [20].

More recently, [21] proposed a method for finding approximate moments of the FPTD for the integrated Brownian process with \( x_0 = 0, v_0 > 0, a \geq 0 \) by the help of martingale methods, stopping-time arguments, and a power series approach.

2) Approximation Methods: As first-passage time problems are generally hard to solve analytically, much effort has been invested in the development of approximation methods. However, these methods are mostly designed for one-dimensional, stationary Gaussian processes. We only point out a few directions here and refer the reader to the surveys [22], [23]. One possibility is to make use of the expected number of up-crossings and express the desired probability by means of a series, often called Rice series [24]. Another approximation, especially popular in physics, is to assume that the time intervals between crossings are independent, known as the independent interval approximation (see, e.g., [5]). Another class of approximation schemes transfers the original problem to a first-passage time problem of a Wiener process with a curved boundary and approximates it by piecewise linear functions [25]. For example, [4] used this approach in combination with a series expansion to approximate the FPTD for Gauß–Markov processes.

IV. GAUSS–TAYLOR APPROXIMATION

We now present our first approach that uses error propagation in combination with a simple Gaussian approximation for the FPTD of LGSSMs. It consists of the steps:

1) Calculate \( \theta_d \) as the first-passage time of the underlying deterministic motion model, i.e., \( \theta_d \) is the solution of the equation \( \mathbb{E}\{x(t)\} = a \) for \( t, \theta_d \) will yield the mean of our Gaussian approximation for the FPTD.

2) Set up the motion equation

\[
\mathbf{z}(t) = \boldsymbol{\Phi}(t-t_0)\mathbf{z}(t_0)\big|_{t_0=\theta_d, t=\mathcal{T}_a, x(\theta_d)=a}
\]

at this point and solve it for \( \mathcal{T}_a \) as a function of \( \mathbf{z}(\theta_d) \).

3) Linearize \( T_a(\mathbf{z}(\theta_d)) \) at \( \mathbb{E}\{\mathbf{z}(\theta_d)\} \) with a first order Taylor series expansion. This yields \( T_a(\mathbf{z}(\theta_d)) \approx \mathbb{E}\{T_a(\mathbf{z}(\theta_d))\}
\]

\[
= T_a(\mathbf{z}(\theta_d))|_{\mathbf{z}(\theta_d)=\mathbb{E}\{\mathbf{z}(\theta_d)\}} + \nabla T_a(\mathbf{z}(\theta_d))|_{\mathbf{z}(\theta_d)=\mathbb{E}\{\mathbf{z}(\theta_d)\}} \cdot (\mathbf{z}(\theta_d) - \mathbb{E}\{\mathbf{z}(\theta_d)\})
\]

4) Determine the variance of the linearization and employ it as the variance of our Gaussian approximation for the FPTD.

All steps combined result in the approximations \( \mathbb{E}\{\mathcal{T}_a\} \approx \theta_d \) and \( \text{Var}\{\mathcal{T}_a\} = \nabla \mathcal{T}_a(\hat{\theta}_d)^\top \cdot \text{Cov}\{\mathbf{z}(\theta_d)\} \cdot \nabla \mathcal{T}_a(\hat{\theta}_d) \).

Example 1 (CV Model, continued). The mean of our approximation for the CV model is given by \( \theta_d = t_0 + (a - \hat{x}^{h_n}) / \hat{x}^{h_n} \).

The mean in the vicinity of \( \theta_d \) is

\[
\mathcal{T}_a(\mathbf{z}(\theta_d), \hat{x}(\theta_d)) = \theta_d + \frac{a - \hat{x}(\theta_d)}{\hat{x}(\theta_d)}
\]

The linearization yields \( \mathcal{T}_a(x(\theta_d), \hat{x}(\theta_d)) \approx \theta_d + \frac{a - x(\theta_d)}{\hat{x}(\theta_d)} \).

Finally, the error propagation results in \( \text{Var}\{\mathcal{T}_a\} \approx \frac{\text{Var}\{x(\theta_d)\}}{\hat{x}(\theta_d)^2} \).

Note that in addition to the approximations by the Gaussian assumption and the linearization, this approach includes another approximation: When setting up the motion equation in the vicinity of \( \theta_d \), we neglect additional noise stemming from the particular solution (second term) of (2) in the time interval between \( \theta_d \) and \( \mathcal{T}_a \). Moreover, its only relation to first-passage time problems is the use of \( \theta_d \) as the point of linearization.

V. NO-RETURN APPROXIMATION

Our second proposed approximation relies on the assumption that particles are unlikely to move back once they have passed the boundary. For this, in the first part, we formalize the requirements, specify the assumption, and present our approximation. In the second part, we show how we can obtain an expression for the quantile function. Third, we propose a method for fast numerical calculation of the moments. Finally, we introduce a method for one or multidimensional Markov processes that can be used to spot-check the initial assumption.

A. Approximate First-Passage Time Probabilities

Let \( x(t) \) be a Gaussian process and \( a \in \mathbb{R} \) a fixed boundary with \( a > x(t_0) \) almost surely, so that \( 3\varepsilon \in (t_0, \infty) \) with

1) \( \mathbb{P}(x(t) > a) \) a monotonically increasing function in \([t_0, t_ɛ]\),

2) \( \mathbb{P}(x(t_ɛ) > a) \approx 1 \), with tolerance \( \varepsilon \), and

3) \( \mathbb{P}(T_a < t, x(t) \leq a) \approx 0 \), \( \forall t \in [t_0, t_ɛ) \).

The second requirement ensures that the first passage is an almost sure event since the true first-passage time probability for \( t \to \infty \) is higher than \( \mathbb{P}(x(t_ɛ) > a) \). The last point encodes our assumption that particles are unlikely to return to a level lower than \( a \) within the interval \( [t_0, t_ɛ) \). Because of the symmetry of a Gaussian process, all three requirements together can be read such that the mean is primarily responsible for the first passage.
For the median, i.e.,

\[ \text{Var}(x(t)) \sim \phi \left( \frac{E(x(t)) - a}{\sqrt{\text{Var}(x(t))}} \right) \quad t < t_c \]

\[ \text{Var}(x(t_c)) = \alpha \quad t \geq t_c \]  

Taking the derivative with respect to \( t \) for \( t < t_c \) yields an approximation for the FPTD

\[
p_{\tau_c}(t) = \left( \frac{d}{dt} E(x(t)) \right) \left( \frac{a - E(x(t))}{2 \sqrt{\text{Var}(x(t))}} \right) \exp \left( \frac{-(a - E(x(t)))^2}{2 \text{Var}(x(t))} \right) .
\]  

(8)

Thus, the median of our approximation corresponds to solving the process’ deterministic counterpart \( E(x(t)) = a \) for \( t \).

C. Moments

For computing the moment of order \( m \), in general, it is necessary to evaluate the Riemann integral \( \int_{t_0}^\infty t^m \rho_{\tau_c}(t) \, dt \), where \( \rho_{\tau_c}(t) \) is the approximate FPTD according to (8). For the CV or CA model, this integration cannot be performed in closed form. Moreover, we have to pay attention to the integration limits, which have to be restricted to \([t_0, t_c]\) in order to avoid divergent integrals.

With these restrictions, it is possible to compute the moments numerically. For this purpose, we propose to use the CDF to accurately measure the actual probability mass in each sub-interval \([t_j, t_{j+1}]\), resulting in the approximation

\[
E(T^m_a)_{[t_a, t_b]} \approx \sum_{j=0}^{n-1} t_j^m \left( \mathbb{P}(T_a < t_{j+1}) - \mathbb{P}(T_a < t_j) \right),
\]

with integration limits \( t_a, t_b \in \mathbb{R} \), integration points \( t_j = t_a + j \Delta t, \Delta t = \frac{t_b - t_a}{n}, \) and a sufficiently large \( n \in \mathbb{N} \). Note that we could also approximate the moments using upper sums \((t_j^m)\) instead of lower sums \((t_j^m)\), and are thus directly provided with a natural measure of integration error by comparing both. It remains to find sufficient integration limits \( t_a \) and \( t_b \). We can use the PPF to find the locations where the probability mass is located, rather than just integrating from \( t_0 \) to \( t_c \), to construct an efficient algorithm for finding the moments.

D. Estimate of Returning Probabilities

We propose a fast-to-calculate method that indicates whether we can assume that particles are unlikely to return to the boundary after a crossing. Therefore, we calculate a time interval for which the process is highly certain to remain below the limit once it has reached it.

In the following, suppose we know the time \( \theta \) of a crossing, i.e., \( x(\theta) = a \). The probability of returning below \( a \) at time \( \theta + \tau \) after a crossing at time \( \theta \), \( \mathbb{P}(x(\theta + \tau) < a | x(\theta)) \), is then given by

\[
\int_{-\infty}^a p(x(\theta + \tau) | x(\theta)) \, dx(\theta + \tau) .
\]  

(9)

For multivariate Markov processes, we can find an expression for \( p(x(\theta + \tau) | x(\theta)) \) by writing the (full) state \( x(\theta) = [x(\theta), \mathbb{F}_R(\theta)]^T \) as a combination of the first state component \( x(\theta) \) and all remaining state components \( \mathbb{F}_R(\theta) \). Then, \( p(x(\theta + \tau) | x(\theta)) \) is given by

\[
\Omega_{\mathbb{F}_R(\theta + \tau)} \Omega_{\mathbb{F}_R(\theta)} \cdot p(x(\theta) | x(\theta)) \, d\mathbb{F}_R(\theta) \, d\mathbb{F}_R(\theta + \tau),
\]

where \( \Omega_{\mathbb{F}_R(\theta)} \) denotes the domain of \( \mathbb{F}_R(\theta) \). As all densities involved are Gaussians, the integration can be performed in closed form (for LGSSMs, see Appendix A).
We can now study the asymptotic behavior of \( \Pr(x(\theta + \tau) < a|x(\theta)) \). When \( \lim_{\tau \to \infty} \sqrt{\text{Var}[x(\theta + \tau)|x(\theta)]} \to a \), that is, the conditional mean increases faster than the conditional standard deviation, the probability of not returning to \( a \) goes to 1. On the other hand, when \( \lim_{\tau \to \infty} \sqrt{\text{Var}[x(\theta + \tau)|x(\theta)]} \to 0 \), the probability of not returning shrinks to \( \frac{a}{2} \) for \( \tau \to \infty \). This implies that returning below the boundary is likely within a (small) time interval after the crossing for a process of the first class (e.g., a Wiener process with drift), whereas for the processes of the second class (e.g., a CV and CA model), the probability of returning increases when waiting long enough.

For finite \( \tau \), the probability \( \Pr(x(\theta + \tau) < a|x(\theta)) \) depends on the time \( \theta \) itself. For an intuition, we could use the deterministic solution for the first-passage time \( \theta_d \), i.e., \( \theta_d \) is set to the solution of \( E[x(t)] = a \) for \( t \). We now can again find the PPF of \( x(\theta_d + \tau) < a|x(\theta_d) \) or use a graphical method to plot the probability over the length of the time interval. If the probability falls below a desired confidence \( q \) within a time relevant to our technical process, this is a strong indicator that condition 3) is invalid and one may consider changing the technical process or the model describing it.

**Example 1** (CV Model, continued). For the CV model, the process mean is given by \( E[x(t)] = \bar{x}^0 + \dot{x}^0 (t - t_0) \) and the variance by \( \text{Var}[x(t)] = \Sigma_x + 2 \dot{x}^0 \Sigma_{\dot{x}x} (t - t_0) + \Sigma_{\dot{x}x} (t - t_0)^2 + S(t-t_0)^3 \). Thus, \( \frac{d}{dt} E[x(t)] = \dot{x}^0 + 2 \dot{x}^0 \Sigma_{\dot{x}x} (t - t_0) \) and \( \frac{d}{dt} \text{Var}[x(t)] = 2 \dot{x}^0 \Sigma_{\dot{x}x} + 2 \Sigma_{\dot{x}x} (t - t_0) + S(t-t_0)^2 \). To determine \( t_c \), one needs to solve the cubic polynomial

\[
-1/3S\dot{x}^0 \dot{t}^3 + S(a - \dot{x}^0) \dot{t}^2 + 2 \left( \dot{x}^0 \Sigma_{\dot{x}x} + (a - \dot{x}^0) \Sigma_{\dot{x}x} \right) \dot{t} + 2 \left( \dot{x}^0 \Sigma_{\dot{x}x} + (a - \dot{x}^0) \Sigma_{\dot{x}x} \right) = 0
\]

with \( \dot{t} = t - t_0 \) for \( t \). Likewise, the calculation of the PPF results in

\[
1/3S \dot{q}^2 \dot{t}^3 + \left( \Sigma_{\dot{x}x} q^2 - \dot{x}^0 q^2 \right) \dot{t}^2 + 2 \left( \Sigma_{\dot{x}x} q^2 + (a - \dot{x}^0) \dot{x}^0 \right) \dot{t} + \Sigma_{\dot{x}x} q^2 - (a - \dot{x}^0)^2 = 0
\]

In particular, the median of our approximation is \( t_0 + \frac{a - \dot{x}^0}{\dot{x}^0} \).

The plots show the FPTDs and corresponding CDFs for our approaches. The histograms illustrate the results of the Monte Carlo approach. Vertical lines visualize the mean, variance, and the first and third quantiles of the methods, respectively. The black solid vertical lines illustrate the solutions of the deterministic model.

**A. Processes Used for the Evaluations**

For all evaluations, we chose the following common parameters taken from a typical particle from our particle sorting application. Particles start at \( \dot{x}_t = 29 \text{mm} \) with velocity \( \dot{x}_t = 600 \text{mm/s} \) at time \( t_0 = 0 \). The boundary is located at \( a = 62.5 \text{mm} \). For the CA model, the acceleration is \( \ddot{x}_t = 425 \text{mm/s}^2 \).

**B. Monte Carlo Simulations**

We first discretize the continuous-time motion models with a sufficiently small time-increment \( \Delta t \) using the analytic solution (2) as described in [7, Chapter 6.2]. We then let each particle run until it has crossed the boundary \( a \) for the first time. Next, we compute the first-passage time using linear interpolation between the last position before and the first position after the boundary. Finally, we use histograms with 100 equally-distributed bins in the plot range to illustrate the distribution. The histograms also take into account particles that have not reached the boundary within the maximum number of time steps used, but these particles are not displayed explicitly. For all of our evaluations, we used 100 000 particles and a time-increment of 0.001 s.

**C. Results**

The plots show the FPTDs and corresponding CDFs for our approaches. The histograms illustrate the results of the Monte Carlo approach. Vertical lines visualize the mean, variance, and the first and third quantiles of the methods, respectively. The black solid vertical lines illustrate the solutions of the deterministic model.

Fig. 3 shows the results of our method for three CV models with increasing power spectral density. For relatively low noise, i.e., for processes with a short-time behavior dominated by the mean, our methods are very close to the results of the Monte Carlo simulation. For low noise levels, the Gauß–Taylor approximation works surprisingly well. However, for higher noise levels, it fails to capture the distribution’s shape correctly, while the No-return approximation still yields good results. In general, compared with the Gauß–Taylor approximation, the No-return approximation is more accurate, since it, e.g., also captures the distribution’s skewness and kurtosis.

Except for the process displayed in Fig. 3c, which has an \( \epsilon \approx 0.29 \), all considered processes meet the requirements 1) and 2), i.e., \( \Pr(x(t) > a) \) approximately reaches 1 while increasing monotonically. For the latter process, also depicted in slightly transparent color, one can observe that \( \Pr(x(t) > a) \) decreases for \( t > t_c \approx 0.104 \) s. For the process in Fig. 3b, \( \epsilon \approx 2 \times 10^{-8} \).

For all other processes, \( \epsilon \) is too small to be calculated. The same qualitative behavior as for CV models can be observed for CA models. We refer to Fig. 1 for an example with a power spectral density of \( S = 9360 \text{445mm}^2/\text{s}^3 \).

Fig. 4 visualizes the returning probability for \( \theta = \theta_d \) for the CV model displayed in Fig. 3a. Here, to achieve a confidence of not returning higher than 95\%, the time difference \( \tau \) after a crossing at \( \theta_d \) must be smaller than 42.58 s. This means the technical process and its model should be such that returning
after $\tau$ is impossible or irrelevant in order for the No-return approximation to be applicable. For the CV model with $S = 93 640 \text{ mm}^2/\text{s}^3$, $\tau$ must be smaller than 4.2 s, and for the CV model with $S = 936 400 \text{ mm}^2/\text{s}^3$, $\tau < 0.000 77 s$ must hold. The calculation times of our Python implementation averaged over 10 runs are compared in Tab. I.\footnote{Evaluated with an Intel Core i7-6700HQ CPU, 8 × 2.60 GHz and an NVIDIA RTX 2080 Ti.}

\begin{table}[h]
\centering
\caption{Comparison of computational times in ms.}
\begin{tabular}{lccc}
\hline
 & MC simulation & Gauß–Taylor & No-return \\
\hline
PDF & 2774 & 0.318 & 10.066 \\
CDF & 3153 & 1.156 & 3.53 \\
PPF & 2971 & 0.361 & 3.727 \\
$E\{T_\alpha\}$ & 2733 & 0.051 & 5.084 \\
$\operatorname{Var}\{T_\alpha\}$ & 2774 & 0.056 & 6.129 \\
\hline
\end{tabular}
\end{table}

We proposed two methods for approximating first-passage time probabilities of Gaussian processes with an increasing trend, such as CV and CA models. The first method yields a simple Gaussian approximation by an error propagation approach. For the second method, we showed how the assumption that particles are unlikely to return to a level lower...
than the boundary once they have crossed it can be used to obtain an analytic approximation for the first-passage time CDF. We then derived an expression for the PDF and its quantile function that only depends on the standard Gaussian PPF. Furthermore, we proposed a numerical method based on Riemann sums to compute the first-passage time moments. While the first approach yields reasonable results for low noise levels, empirical evaluations for the CV and the CA model prove the validity and robustness of the second method. It is thus particularly suitable for motion models from engineering, also in terms of required computation time.

### Appendix A

**Transition Density of LGSSMs Conditioned on One State Component**

Our goal is to evaluate $p(x(\theta + \tau)|x(\theta))$,

$$p(x(\theta + \tau)|x(\theta), \mathbb{I}_R(\theta + \tau)|x(\theta), \mathbb{I}_R(\theta)) \cdot p(\mathbb{I}_R(\theta)|x(\theta)) \, dx_R(\theta) \, dx_R(\theta + \tau),$$

where all densities involved are from an LGSSM. Furthermore, we have written the (full) state $\mathbb{I}(\theta) = [x(\theta), \mathbb{I}_R(\theta)]^T$ as a combination of the first state component $x(\theta)$ and all remaining state components $\mathbb{I}_R(\theta)$. Because $\mathbb{I}(\theta)$ is jointly Gaussian,

$$p(\mathbb{I}_R(\theta)|x(\theta)) = \mathcal{N}(x(\theta) - \hat{x}(\theta), \Sigma_{\mathbb{I}_R(\theta)})$$

and covariance

$$\text{Cov}(\mathbb{I}_R(\theta)|x(\theta)) = \Sigma_{\mathbb{I}_R(\theta)} - \Sigma_{\mathbb{I}_R(\theta)} \Sigma_{x(\theta)} \Sigma_{\mathbb{I}_R(\theta)}.$$ 

Here, we used $\Sigma_{\mathbb{I}_R(\theta)}$, $\Sigma_{x(\theta)}$, and $\Sigma_{\mathbb{I}_R(\theta)}$ to denote the respective entries of $\text{Cov}(\mathbb{I}(\theta))$. The LGSSM’s transition density $p(\mathbb{I}(\theta + \tau)|x(\theta))$ is given by the Gaussian process with mean $\Phi(\tau) x(\theta)$ and covariance

$$\text{Cov}(\mathbb{I}(\theta + \tau)|x(\theta)) = \int_0^\tau \Phi(t) \text{D}V(t)^T \Phi(t)^T \, dt.$$ 

Now, evaluating the inner integral of (10), incorporating that

$$\mathbb{I}(\theta) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} x(\theta) + \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \mathbb{I}_R(\theta)$$ 

and exploiting that both processes are Gaussian, yields $p(\mathbb{I}(\theta + \tau)|x(\theta))$, which again is a Gaussian process with mean and covariance

$$\text{E}(\mathbb{I}(\theta + \tau)|x(\theta)) = \Phi(\tau) \mathbb{I}_R(\theta)$$

$$\text{Cov}(\mathbb{I}(\theta + \tau)|x(\theta)) = \Phi(\tau) \Sigma_{\mathbb{I}_R(\theta)} \Phi(\tau)^T.$$ 

$p(x(\theta + \tau)|x(\theta))$ is the first component of $p(\mathbb{I}(\theta + \tau)|x(\theta))$. 

### References


