CLOSED–FORM ELLIPTIC LOCATION WITH AN ARBITRARY ARRAY TOPOLOGY

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ABSTRACT

An efficient noniterative algorithm for active localization of objects is developed. It is based on intersecting elliptic curves defined by uncertain range–sum measurements between a signal source, the objects, and a number of arbitrarily located receivers. Measurement errors are modeled as being unknown but bounded in amplitude by a closed convex set. Based on this set–theoretic uncertainty model, an error propagation analysis is performed, that allows to accurately bound estimation errors. For discriminating object primitives and for discarding erroneous measurements, a hypothesis test is derived. The algorithm's computational load is much lower than for grid–based methods and iterative techniques. A recursive formulation is provided to support real–time applications.

1. INTRODUCTION

Much of the existing literature is devoted to the problem of passively locating signal sources. In many applications, however, it is of interest to illuminate a non–radiating scene and determine object positions based on range measurements. For one signal emitter and a linear array of receivers, closed–form solutions are given in [1], [2], [3].

A more complex situation is encountered for N arbitrarily located sensors, which receive the signal emitted by a detached source and reflected from distinct, solid objects. Such a configuration allows the treatment of hull–mounted conformal arrays, which are often found in practical location systems. Intersecting the elliptic curves which describe the solution loci, is done numerically on a grid in [4], resulting in a high computational load. In addition, grid–based schemes are not very efficient for the case of a few distinct reflectors. Analytic solutions are often based on a far–field approximation, that may not be accurate enough for certain applications. The straightforward least–squares solution of the exact analytic equations requires a costly iterative minimization.

This paper is concerned with *efficient* algorithms for locating object primitives with an *arbitrary array topology.* Based on ideas for passive source location from range–difference measurements reported in [5], [6], [7], the non–linear measurement equations are transformed to an error vector that may be minimized in closed form in a weighted least–squares sense.

A–priori knowledge on the measurement uncertainties is modeled in a set–theoretic framework, i.e., ranging errors are assumed to be unknown but bounded in amplitude. This is a useful alternative to stochastic models when coping with real–world phenomena like correlated noise, unknown statistics, systematic errors, and errors which are not mutually independent for different sensors. Based on this uncertainty model, an error propagation analysis is performed. A weighting matrix for the least–squares estimator is derived, that is optimal, when the ranging errors are within practical limits. For discrimination of different objects and for discarding erroneous measurements, a set–theoretic hypothesis test is provided.

The efficient location estimator is presented in Sec. 2. A recursive formulation of the proposed algorithm is given in Sec. 3. Simulation results shown in Sec. 4 demonstrate the effectiveness of the proposed location algorithm. An application to obstacle detection and location with the multisonar system of a mobile robot may be found in [8].

2. EFFICIENT LOCATION ESTIMATOR

N omnidirectional receivers at arbitrary but known positions $\underline{C}_i = [C_i^x, C_i^y, C_i^z]$, $i = 1, 2, \ldots, N$, are considered. A signal source located at the origin emits a signal that is reflected by illuminated objects. Objects are modeled as points or as specular reflecting planes similar to the discussion in [9]. Measurements R_i are obtained as the sum of the ranges source $\rightarrow$ object and object $\rightarrow$ receiver i. Let us first consider the measurement model for a point reflector depicted in Fig. 1 a).

2.1. Point Reflector

Let $\underline{\psi} = [\psi^x, \psi^y, \psi^z]^T$ and $r = \|\underline{\psi}\|$ denote the object position and distance respectively. From the figure it follows

$$(R_i - r)^2 = \|\underline{\psi} - \underline{C}_i\|^2 \,, \; i = 1, 2, \ldots, N \,. \qquad (1)$$

For $N > 3$ this is an overdetermined system of equations. Given a noisy measurement vector $\underline{R} = [R_1, R_2, \ldots, R_N]^T$, it would be desirable to select the estimated object position $\hat{\underline{\psi}}$ such that some norm of the differences between the masurement vector $\underline{R}$ and the vector $\hat{\underline{R}}$ of ranges given $\hat{\underline{\psi}}$ is minimized. However, even for a weighted L_2 norm this approach calls for an iterative minimization, since $\hat{\underline{R}}$ is a nonlinear function of $\hat{\underline{\psi}}$.

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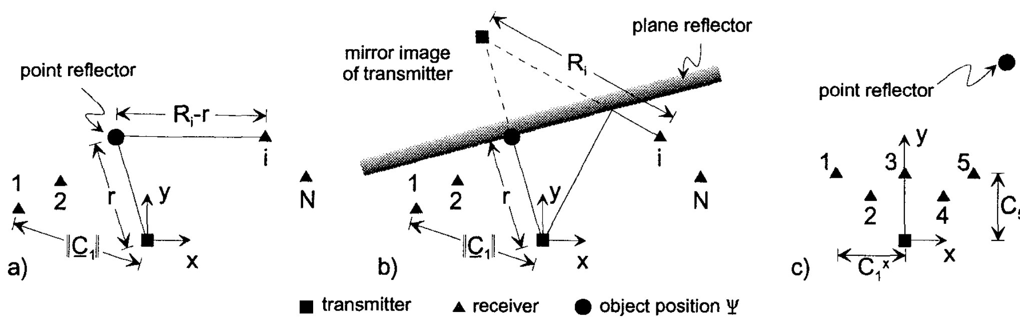

Figure 1. Measurement model for a) point reflector, b) plane reflector, c) simulations in Sec. 4.

Inspired by the treatment of hyperbolic location in [5], an error vector $\underline{e}$ is derived from (1), given by

$$\underline{e} = \underline{\Delta} + 2r\underline{R} - \mathbf{H}\underline{\psi} \tag{2}$$

with

$$\underline{\Delta} = \begin{bmatrix} \|\underline{C}_1\|^2 - R_1^2 \\ \|\underline{C}_2\|^2 - R_2^2 \\ \vdots \\ \|\underline{C}_N\|^2 - R_N^2 \end{bmatrix}, \quad \mathbf{H} = 2\begin{bmatrix} \underline{C}_1^T \\ \underline{C}_2^T \\ \vdots \\ \underline{C}_N^T \end{bmatrix}, \tag{3}$$

which may be minimized in closed-form. Before we obtain a position estimate from (2), r has to be eliminated. For this purpose, the least-squares solution of $\underline{\psi}$ in terms of r is written as

$$\underline{\psi} = \underline{\alpha} + 2r\underline{\beta} \tag{4}$$

with

$$\underline{\alpha} = \mathbf{G}\underline{\Delta}\,, \ \underline{\beta} = \mathbf{G}\underline{R} \tag{5}$$

and

$$\mathbf{G} = \left(\mathbf{H}^T\mathbf{W}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{W}\,, \tag{6}$$

where $\mathbf{W}$ is a weighting matrix. Similar to ideas in [6], [7] for range–difference location, the relationship $r = \underline{\psi}^T\underline{\psi}$ is exploited to obtain a quadratic equation for r given by

$$(4\underline{\beta}^T\underline{\beta} - 1)r^2 + 4\underline{\alpha}^T\underline{\beta}r + \underline{\alpha}^T\underline{\alpha} = 0\,. \tag{7}$$

A positive root [1] $\hat{r}$ of (7) now serves as an estimate for r in (2) and a weighted least-squares estimate $\underline{\hat{\psi}}$ for the object position can be obtained.

Before expressions for the weighting matrices are derived, some assumptions concerning the measurement errors have to be made. Noise free values of $*$ are denoted as $\tilde{*}$ in the sequel. The true range vector $\underline{\tilde{R}}$ is assumed to be corrupted by additive errors during the measurement process according to

$$\underline{R} = \underline{\tilde{R}} + \underline{\Sigma}\,. \tag{8}$$

Furthermore, the measurement errors are assumed to be unknown but bounded by an ellipsoid, i.e.,

$$\underline{\Sigma}^T\mathbf{S}_\Sigma^{-1}\underline{\Sigma} \le 1\,, \tag{9}$$

where $\mathbf{S}_\Sigma$ is a symmetric, positive definite matrix.

[1] In the case of two positive roots, the ambiguity must be resolved by incorporating additional information.

It can be proved [8], that the optimal weighting matrix $\mathbf{W}$ in (6) is given by

$$\mathbf{W} = \{(\tilde{\mathbf{R}}_D + 2\tilde{r}\mathbf{I})\mathbf{S}_\Sigma(\tilde{\mathbf{R}}_D + 2\tilde{r}\mathbf{I})\}^{-1}\,. \tag{10}$$

After inserting a solution $\hat{r}$ from (7) into (2), the error $\underline{e}$ is bounded by an ellipsoid $\underline{e}^T\mathbf{E}^{-1}\underline{e} \le 1$ for "small" errors $\underline{\Sigma}$. $\mathbf{E}$ is given by $\mathbf{E} = \mathbf{D}\mathbf{S}_\Sigma\mathbf{D}^T$, where

$$\mathbf{D} = \left\{\mathbf{I} - \underline{\tilde{R}}\,\underline{\tilde{\gamma}}^T\mathbf{G}\right\}(\tilde{\mathbf{R}}_D + 2\tilde{r}\mathbf{I}) \tag{11}$$

with

$$\mathbf{R}_D = \mathrm{diag}[R_1, R_2, \ldots, R_N] \tag{12}$$

and

$$\underline{\tilde{\gamma}} = 2\left(2\underline{\tilde{\psi}}^T\underline{\tilde{\beta}} - \tilde{r}\right)^{-1}\underline{\tilde{\psi}}\,. \tag{13}$$

The optimal weighting matrix for the least–squares estimate of the object position given an estimate $\hat{r}$ is thus $\mathbf{E}^{-1}$.

The matrices $\mathbf{W}$, $\mathbf{D}$ contain true quantities, that are of course unknown. Some approximations are therefore necessary. The measured distances are used to approximate the unknown true distance vector $\underline{\tilde{R}}$. In a first step, $\{\mathbf{R}_D\mathbf{S}_\Sigma\mathbf{R}_D\}^{-1}$ is used instead of $\mathbf{W}$ to solve for $\underline{\alpha}$, $\underline{\beta}$. Once $\hat{r}$ is calculated, it may serve to approximate $\tilde{r}$ in (10). When calculating the least–squares estimate of the object position with (2) given $\hat{r}$, in a first step $\mathbf{W}$ is used to approximate $\mathbf{E}^{-1}$. The obtained estimated values are then used as an approximation of the true values in (11) to calculate $\mathbf{D}$. The described process could be iterated to provide even better solutions. However, this does not lead to significant changes in practical situations.

The set of all point reflector positions compatible with the error bounds is now given as an ellipsoidal set [10]

$$\Omega = \{\underline{\psi} : (\underline{\psi} - \underline{\hat{\psi}})^T\mathbf{S}_\psi^{-1}(\underline{\psi} - \underline{\hat{\psi}}) \le 1\}\,, \tag{14}$$

where

$$\mathbf{S}_\psi = d_{Point}\left(\mathbf{H}^T\mathbf{E}^{-1}\mathbf{H}\right)^{-1}\,. \tag{15}$$

d_{Point} is given by

$$d_{Point} = 1 - \underline{e}^T(\underline{\hat{\psi}})\mathbf{E}^{-1}\underline{e}(\underline{\hat{\psi}}) \tag{16}$$

and represents a measure for the distance of the actual error from the surface of the error ellipsoid defined by $\mathbf{E}$. When d_{Point} is negative, the solution is not compatible with the error bounds. For $0 < d_{Point} \le 1$, the solution is valid and given as the ellipsoid defined by $\mathbf{S}_\psi$ and centered at $\underline{\hat{\psi}}$.

2.2. Plane Reflector

The measurement equation for a plane reflector is obtained from Fig. 1 b) as

$$R_i^2 = \|\underline{C}_i - 2\underline{\psi}\|^2 , \; i = 1, 2, \ldots, N . \qquad (17)$$

The error vector $\underline{e}$ is written as

$$\underline{e} = \underline{\Delta} + 4\, r^2\, \underline{1} - \mathbf{H}\underline{\psi} \qquad (18)$$

with

$$\underline{\Delta} = \begin{bmatrix} \|\underline{C}_1\|^2 - R_1^2 \\ \|\underline{C}_2\|^2 - R_2^2 \\ \vdots \\ \|\underline{C}_N\|^2 - R_N^2 \end{bmatrix} , \; \mathbf{H} = 4 \begin{bmatrix} \underline{C}_1^T \\ \underline{C}_2^T \\ \vdots \\ \underline{C}_N^T \end{bmatrix} , \; \underline{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} . \qquad (19)$$

Similar considerations as for the point reflector lead to an efficient plane location estimator.

The set–theoretic formulation naturally provides a hypothesis test for the type of reflecting object primitive. The hypothesis "point reflector" will be accepted, if the point condition is true, i.e., if $0 < d_{Point} \leq 1$. In analogy, the hypothesis "plane reflector" will be accepted, if the respective condition is true, i.e., if $0 < d_{Plane} \leq 1$. No decision is possible, if both conditions are true. Both hypothesis are rejected, if no condition is true.

3. RECURSIVE FORMULATION OF THE EFFICIENT ESTIMATOR

In many applications it is advantageous to update position estimates by sequential inclusion of measurements from different sensors. This section is concerned with the derivation of a recursive version of the efficient batch location estimator derived in Sec. 2. We restrict attention to the point reflector model; similar arguments hold for a plane reflector.

The derivation of a recursive formulation is complicated by the fact that $\mathbf{E}$ is not a diagonal matrix even if $\mathbf{S}_\Sigma$ is of diagonal type. However, the inverse of $\mathbf{D}$ is given by

$$\mathbf{D}^{-1} = (\tilde{\mathbf{R}}_D + 2\tilde{r}\mathbf{I})^{-1}\{\mathbf{I} + \underline{\tilde{R}}\underline{\tilde{\eta}}^T\mathbf{G}\} \qquad (20)$$

with

$$\underline{\tilde{\eta}} = \left[1 - \underline{\tilde{\gamma}}^T\underline{\tilde{\beta}}\right]^{-1} \underline{\tilde{\gamma}} . \qquad (21)$$

For $\mathbf{S}_\Sigma = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2)$, (20) is used to convert the problem of estimating the object position with (2) given an estimated object distance $\hat{r}$ from (7) to a standard weighted recursive least–squares estimator

$$\begin{aligned} \underline{\alpha}_k &= \underline{\alpha}_{k-1} + w_k\mathbf{P}_k\underline{h}_k^*[\Delta_k^* - \underline{h}_k^{*T}\underline{\alpha}_{k-1}] \\ \underline{\beta}_k &= \underline{\beta}_{k-1} + w_k\mathbf{P}_k\underline{h}_k^*[R_k^* - \underline{h}_k^{*T}\underline{\beta}_{k-1}] \\ \mathbf{P}_k &= \mathbf{P}_{k-1} - \frac{\mathbf{P}_{k-1}\underline{h}_k^*\underline{h}_k^{*T}\mathbf{P}_{k-1}}{\underline{h}_k^{*T}\mathbf{P}_{k-1}\underline{h}_k^* + w_k} \end{aligned} \qquad (22)$$

for $k = 1, 2, \ldots, N$, with $w_k = (4(\tilde{r} - \tilde{R}_k)^2\sigma_k^2)^{-1}$, $\underline{\alpha}_0 = (0,0)^T$, $\underline{\beta}_0 = (0,0)^T$, $\mathbf{P}_0 = \epsilon\mathbf{I}$, $\epsilon\ldots$large, and the transformations

$$\begin{aligned} \underline{h}_k^* &= 2\underline{C}_k + \tilde{R}_k\underline{\tilde{\eta}} \\ \Delta_k^* &= \Delta_k + \tilde{R}_k\underline{\tilde{\eta}}^T\underline{\tilde{\alpha}} \\ R_k^* &= \tilde{R}_k + \tilde{R}_k\underline{\tilde{\eta}}^T\underline{\tilde{\beta}} . \end{aligned} \qquad (23)$$

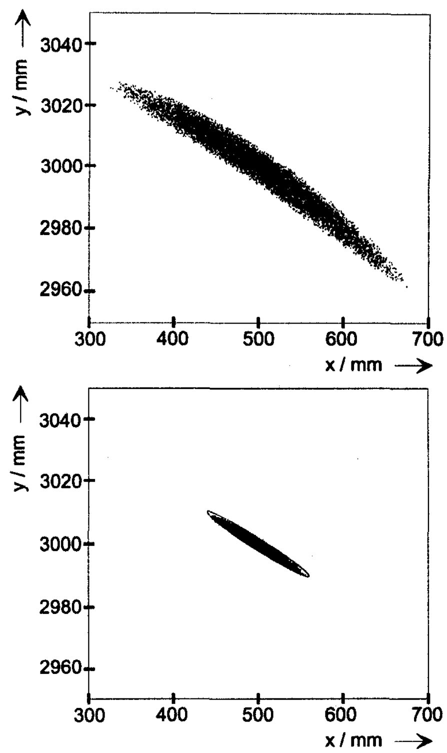

Figure 2. top: Unweighted position estimates, bottom: Prior knowledge on noise bounds exploited.

$\mathbf{S}_\psi$ is calculated from $\mathbf{P}_N = \left(\mathbf{H}^T\mathbf{E}^{-1}\mathbf{H}\right)^{-1}$ and a recursive formulation of d_{Point} given by

$$\begin{aligned} d_{Point,k} &= 1 - \mathsf{A}_k - 4\hat{r}_k\mathsf{B}_k + 2\mathsf{C}_k\underline{\hat{\psi}} - 4\hat{r}_k^2\mathsf{D}_k \\ &\quad + 4\hat{r}_k\mathsf{E}_k\underline{\hat{\psi}} - \underline{\hat{\psi}}^T\mathsf{F}_k\underline{\hat{\psi}} , \end{aligned} \qquad (24)$$

$k = 1, 2, \ldots, N$ with

$$\begin{aligned} &\mathsf{A}_0 = 0, \;\; \mathsf{A}_k = \mathsf{A}_{k-1} + w_k\Delta_k^2 \\ &\mathsf{B}_0 = 0, \;\; \mathsf{B}_k = \mathsf{B}_{k-1} + w_k\tilde{R}_k\Delta_k \\ &\mathsf{C}_0 = 0, \;\; \mathsf{C}_k = \mathsf{C}_{k-1} + 2w_k\Delta_k\underline{C}_k^T \\ &\mathsf{D}_0 = 0, \;\; \mathsf{D}_k = \mathsf{D}_{k-1} + w_k\tilde{R}_k^2 \\ &\mathsf{E}_0 = 0, \;\; \mathsf{E}_k = \mathsf{E}_{k-1} + 2w_k\tilde{R}_k\underline{C}_k^T \\ &\mathsf{F}_0 = 0, \;\; \mathsf{F}_k = \mathsf{F}_{k-1} + 4w_k\underline{C}_k\underline{C}_k^T . \end{aligned} \qquad (25)$$

In analogy to the batch solutions, the recursion formulae include weighting terms that contain unknown true values. Similar approximations as in Sec. 2 are used.

4. SIMULATIVE VERIFICATION

The performance of the proposed object location algorithm is now demonstrated by Monte–Carlo simulations. The first simulation demonstrates the accuracy improvement obtained when using the optimal weighting matrices compared to unweighted least squares estimates. The second simulation shows that bounds for the estimation errors are reliably predicted even for the high noise case.

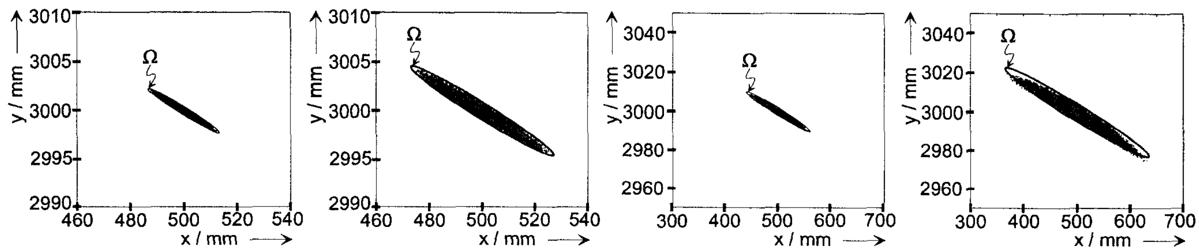

Figure 3. Estimates for different noise levels compared with the predicted set Ω: $k = 0.5$, $k = 2$, $k = 10$, $k = 50$.

The employed measurement topology is depicted in Fig. 1 c). The receivers are located at $\underline{C}_1 = [-300, 300, 0]^T$, $\underline{C}_2 = [-150, 200, 0]^T$, $\underline{C}_3 = [0, 300, 0]^T$, $\underline{C}_4 = [150, 200, 0]^T$, $\underline{C}_5 = [300, 300, 0]^T$. A point reflector is placed at $\underline{\tilde{\psi}} = [500, 3000, 0]^T$ and $\mathbf{S}_\Delta = k \cdot \text{diag}(100, 5, 5, 500, 5)$. For $k = 10$, Fig. 2 compares unweighted least squares estimates with the estimate obtained when using the optimal weighting matrices. The accuracy improvement is obvious.

In Fig. 3 the centers of the resulting estimation sets are plotted for 10000 trials with different noise levels. For comparison purposes, the set Ω is drawn for perfect observations. It may be seen from the figure that Ω provides a reliable bound for the estimation errors even for noise levels that will not be encountered in practical applications.

5. CONCLUSIONS

A simple and efficient closed–form solution for active location estimation has been presented. It is based on range–sum measurements obtained with an arbitrary array topology. The estimator yields more accurate results than grid–based or approximate schemes and offers a lower computational load than grid–based or iterative techniques. Applications include acoustic imaging, seismical sound exploration, and radar.

The measurement errors are modeled as being unknown but bounded in amplitude by a closed convex set. This set–theoretic uncertainty model is particularly useful when the sensors suffer from strongly correlated noise or deterministic errors and when the errors for different sensors are not mutually independent. A rigorous error propagation analysis has been performed. Rather than a point estimate, the estimator outputs a convex set of locations compatible with the a–priori error bounds. As a result, bounds for the estimation errors are accurately predicted for a measurement noise range far beyond practical requirements.

Two object primitives, i.e., points and planes, have been considered and treated in a unified framework. Discriminating these objects given the measured range–sums and the a–priori error bounds is performed via a set–theoretic hypothesis test.

This paper discussed the case of locating one single object. In the presence of multiple unknown objects, a search for the optimum association of measured ranges to objects must be performed. To avoid unnecessary recalculations while testing candidate associations, a recursive formulation of the proposed location algorithm is provided. Furthermore, this recursive solution is useful for real–time applications where measurements are obtained sequentially.

Simulations demonstrate the performance of the proposed efficient location estimator. An application to the problem of detecting and locating unknown obstacles with the multisonar system of a mobile robot may be found in [8].

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