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On Combining Set Theoretic and Bayesian Estimation

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Abstract

We consider state estimation based on observations which are simultaneously corrupted by a deterministic amplitude-bounded unknown bias and a possibly unbounded random process. This problem is solved by developing a combined set theoretic and Bayesian recursive estimator. It provides a continuous transition between both concepts in that it converges to a set theoretic estimator when the stochastic error vanishes and to a Bayesian estimator when the deterministic error vanishes. In the mixed noise case, the new estimator supplies solution sets defined by bounds that are uncertain in a statistical sense.

1 Introduction

Unknown bias terms have found some attention in literature. Several authors employ augmentation of the state space model, [1] uses a bank of Kalman filters. [5] criticizes state augmentation and provides an innovative approach using minimum bias priors based on ignorance. The separate estimation of bias and state has been discussed in [2].

We introduce a new idea for state estimation from observations of several information sources that suffer from two different uncertainties simultaneously. One type of uncertainty is a deterministic but unknown error for which hard amplitude bounds are given a priori. The other type of uncertainty is a stochastic process with given statistics. Prior knowledge on both forms of uncertainty allows conceptionally different reductions in uncertainty during observation of the sample paths of the information sources. The combined **S**tatistical and **S**et theoretic **I**nformation (SSI) filter includes the classical estimation schemes as border cases: It converges to a set theoretic estimator when the stochastic error approaches zero and to a Bayesian estimator when the deterministic error approaches zero. In the mixed noise case, the resulting estimate is a solution set with bounds that are uncertain in a statistical sense. This solution set converges to the intersection of the underlying noise-free sets for an *infinite number* of observations per source. A rigorous problem formulation is given in Sec. 2. The special case of two information sources is discussed in Sec. 3. A recursive SSI filter for an arbitrary number of information sources is introduced in Sec. 4. Numerical solution formulae are given for arbitrary noise densities, simplified solutions arise for the case of Gaussian densities. Numerical examples in the context of mobile robot localization are presented in Sec. 5 to clarify the conveyed concepts. This paper is limited to the scalar case, Sec. 6 provides hints for generalization to higher dimensions.

2 Problem Formulation

N possibly conflicting sources of information S_i, $i = 1, \ldots, N$, on a desired state x are given. Each source S_i is assumed to be corrupted by two types of additive errors. The first error is of deterministic type, i.e., constant and unknown. It is bounded in amplitude by a set. This set is an interval for the scalar case. The second error is represented by a discrete-time, zero-mean, possibly non-white stochastic process SP_i with known statistics. The stochastic processes for different sources are mutually independent. For the scalar case, the measurement equation may be written as[1]

$$Z_i = x + {}^i e_d + {}^i E_s, \; {}^i e_d \in \left[-\frac{b_i}{2}, \frac{b_i}{2}\right], \; {}^i E_s \sim \mathrm{SP}_i \ . \quad (1)$$

When the influence of the stochastic error ${}^iE_s$ is negligible, estimating the state x given observations Z_i is performed by set intersection and results in an interval estimate. The influence of the statistical uncertainty is eventually ruled out by filtering many outcomes of every source S_i, i.e., standard set intersection could be performed after many observations have been aquired. In this paper, we want to go one step further and want to make an estimate of the state x available at every time k. This estimate is of course not a point estimate, but a set estimate where the set bounds are uncertain in a statistical sense. It should asymptotically converge to the above mentioned noise-free interval estimate.

[1] Capital letters are used for random variables or processes, small letters denote specific realizations or deterministic quantities.

3 Two Information Sources

We consider the special case of two independent information sources S_i, $i = 1, 2$. Each source is characterized by a priori knowledge on bounds b_i of the deterministic and constant offset and noise densities f_i^k at time k. In the sequel, we derive the joint density for the left and right bound of the resulting interval estimator. Analytical results are given for marginal densities in case of Gaussian densities f_i^k.

3.1 Arbitrary Noise Densities

$\hat{X}_i^k$ is an estimator of $x + {}^i e_d$ with density $\hat{f}_i^k$. The joint density of the independent random quantities $\hat{X}_i^k$, $i = 1, 2$, is given by $\hat{f}_{12}^k = \hat{f}_1^k \hat{f}_2^k$. The additional prior knowledge on the bounds b_i of the deterministic error ${}^i e_d$ allows elimination of the regions of this joint density for which $\left|\hat{x}_2^k - \hat{x}_1^k\right| > \frac{1}{2}(b_1 + b_2)$ holds. The resulting density $\hat{\hat{f}}_{12}^k$ is then given by

$$\hat{\hat{f}}_{12}^k = \begin{cases} \hat{f}_{12}^k / C & \text{for } \left|\hat{x}_2^k - \hat{x}_1^k\right| \leq \frac{1}{2}(b_1 + b_2) \\ 0 & \text{elsewhere} \end{cases} , \quad (2)$$

where C is a normalizing constant. The left and right bounds ${}^2L^k$ and ${}^2R^k$ of the interval estimator at time k are defined as

$$ {}^2L^k = \max\left(\hat{\hat{X}}_1^k - \frac{b_1}{2}, \hat{\hat{X}}_2^k - \frac{b_2}{2}, \right) , \quad (3)$$

$$ {}^2R^k = \min\left(\hat{\hat{X}}_1^k + \frac{b_1}{2}, \hat{\hat{X}}_2^k + \frac{b_2}{2}, \right) , \quad (4)$$

where $\hat{\hat{X}}_1^k$, $\hat{\hat{X}}_2^k$ are jointly distributed with density $\hat{\hat{f}}_{12}^k$. The joint density of the lower and upper bound of the interval estimator is derived with the aid of Fig. 1 for $b_1 < b_2$ without loss of generality and given by

$$ {}^2f_{LR}^k(l, r) = \frac{1}{{}^2C_{LR}^k} \quad (5)$$

$$\begin{cases} \left[\hat{f}_1^k\left(r - \frac{b_1}{2}\right)\hat{f}_2^k\left(l + \frac{b_2}{2}\right) + \hat{f}_1^k\left(l + \frac{b_1}{2}\right)\hat{f}_2^k\left(r - \frac{b_2}{2}\right) \right. \\ \left. \quad + \delta(l + b_1 - r)\hat{f}_1^k\left(l + \frac{b_1}{2}\right) \int\limits_{x = l + b_1 - \frac{b_2}{2}}^{l + \frac{b_2}{2}} \hat{f}_2^k(x)\, dx \right] \\ \qquad \text{for } l \leq r \leq l + b_1 \\ 0 \qquad \text{elsewhere} \end{cases}$$

where $\delta(x)$ denotes Dirac's impulse function. ${}^2C_{LR}^k$ is a normalizing constant and selected such that $\int\limits_{l=-\infty}^{\infty} \int\limits_{r=-\infty}^{\infty} {}^2f_{LR}^k(l, r)\, dr\, dl$ equals 1. The proof of (5) is trivial and left to the reader.

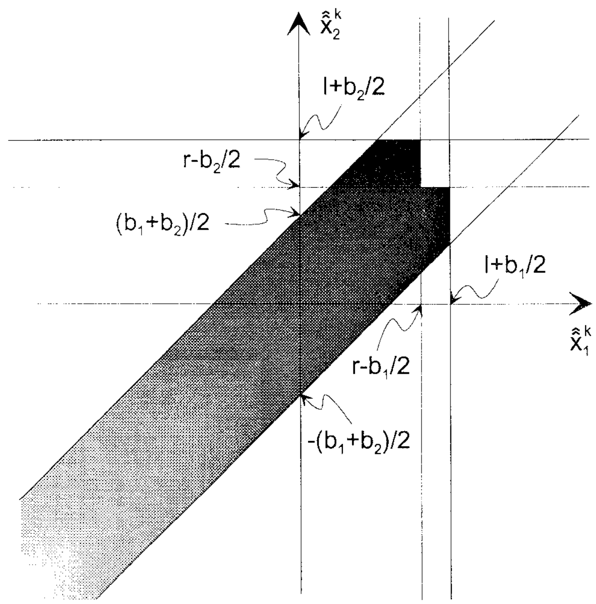

Figure 1: Visualization aid for derivation of the joint distribution ${}^2F_{LR}^k(l, r)$.

3.2 Gaussian Noise Densities

If f_i^k, $i = 1, 2$, are Gaussian, of course $\hat{f}_i^k$, $i = 1, 2$, are also Gaussian with

$$\hat{f}_i^k(x) = \frac{1}{\sqrt{2\pi}\hat{\sigma}_i^k} \exp\left(-\frac{1}{2}\left(\frac{x - \hat{m}_i^k}{\hat{\sigma}_i^k}\right)^2\right) , \quad (6)$$

and analytical expressions for the normalizing constant as well as the marginal densities and expected values of left and right bound, ${}^2L^k$ and ${}^2R^k$, are obtained. The normalizing constant ${}^2C_{LR}^k$ is given by

$$\begin{aligned} {}^2C_{LR}^k = &\operatorname{erf}\left(\frac{\hat{m}_1^k - \hat{m}_2^k + \frac{b_1}{2} + \frac{b_2}{2}}{\sqrt{\left(\hat{\sigma}_1^k\right)^2 + \left(\hat{\sigma}_2^k\right)^2}}\right) \\ &-\operatorname{erf}\left(\frac{\hat{m}_1^k - \hat{m}_2^k - \frac{b_1}{2} + \frac{b_2}{2}}{\sqrt{\left(\hat{\sigma}_1^k\right)^2 + \left(\hat{\sigma}_2^k\right)^2}}\right) \\ &+\operatorname{erf}\left(\frac{\hat{m}_2^k - \hat{m}_1^k + \frac{b_1}{2} + \frac{b_2}{2}}{\sqrt{\left(\hat{\sigma}_1^k\right)^2 + \left(\hat{\sigma}_2^k\right)^2}}\right) \\ &-\operatorname{erf}\left(\frac{\hat{m}_2^k - \hat{m}_1^k + \frac{b_1}{2} - \frac{b_2}{2}}{\sqrt{\left(\hat{\sigma}_1^k\right)^2 + \left(\hat{\sigma}_2^k\right)^2}}\right) , \end{aligned} \quad (7)$$

with the erf–function defined as

$$\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{y=0}^{x} \exp\left(-\frac{y^2}{2}\right) dy \tag{8}$$

according to [8].

The marginal density ${}^2f_L^k(l) = \int\limits_{r=-\infty}^{\infty} f_{LR}^k(l,r)\, dr$ of the lower bound may be expressed as

$$\begin{aligned}{}^2f_L^k(l) = &\frac{1}{{}^2C_{LR}^k}\Bigg[\hat{f}_1^k\left(l+\frac{b_1}{2}\right) \\ &\left(\operatorname{erf}\left(\frac{l+\frac{b_2}{2}-\hat{m}_2^k}{\hat{\sigma}_2^k}\right) - \operatorname{erf}\left(\frac{l-\frac{b_2}{2}-\hat{m}_2^k}{\hat{\sigma}_2^k}\right)\right) \\ &+ \hat{f}_2^k\left(l+\frac{b_2}{2}\right) \\ &\left(\operatorname{erf}\left(\frac{l+\frac{b_1}{2}-\hat{m}_1^k}{\hat{\sigma}_1^k}\right) - \operatorname{erf}\left(\frac{l-\frac{b_1}{2}-\hat{m}_1^k}{\hat{\sigma}_1^k}\right)\right)\Bigg]\end{aligned} \tag{9}$$

The marginal density ${}^2f_R^k(r) = \int\limits_{l=-\infty}^{\infty} f_{LR}^k(l,r)\, dl$ of the upper bound is given by

$$\begin{aligned}{}^2f_R^k(r) = &\frac{1}{{}^2C_{LR}^k}\Bigg[\hat{f}_1^k\left(r-\frac{b_1}{2}\right) \\ &\left(\operatorname{erf}\left(\frac{r+\frac{b_2}{2}-\hat{m}_2^k}{\hat{\sigma}_2^k}\right) - \operatorname{erf}\left(\frac{r-\frac{b_2}{2}-\hat{m}_2^k}{\hat{\sigma}_2^k}\right)\right) \\ &+ \hat{f}_2^k\left(r-\frac{b_2}{2}\right) \\ &\left(\operatorname{erf}\left(\frac{r+\frac{b_1}{2}-\hat{m}_1^k}{\hat{\sigma}_1^k}\right) - \operatorname{erf}\left(\frac{r-\frac{b_1}{2}-\hat{m}_1^k}{\hat{\sigma}_1^k}\right)\right)\Bigg]\end{aligned} \tag{10}$$

Due to space limitations, the closed–form solutions for the expected values of upper and lower bound are omitted.

4 N Information Sources

The above insights are now generalized to the case of N information sources. We focus attention to the derivation of the marginal densities for the upper and lower bound, since they are of major interest for practical applications.

4.1 Arbitrary Noise Densities

We begin with the derivation of a source–recursive expression for the marginal density of the lower bound denoted by ${}^jL^k$, which contains the information from sources $1, \ldots, j$, and is defined by

$${}^jL^k = \max\left({}^{j-1}L^k,\, \hat{X}_j^k - \frac{b_j}{2}\right) \;. \tag{11}$$

The prior knowledge on the deterministic uncertainty bound b_j may be used to formulate the following inequalities for specific realizations $\hat{x}_j^k$, ${}^{j-1}l^k$, ${}^{j-1}r^k$

$$\hat{x}_j^k - \frac{b_j}{2} \le {}^{j-1}r^k, \quad \hat{x}_j^k + \frac{b_j}{2} \ge {}^{j-1}l^k \;. \tag{12}$$

With (11) and (12), we obtain

$$\begin{aligned}{}^jf_L^k({}^jl_k) = &\frac{1}{{}^jC_L^k} \\ &\frac{\partial}{\partial\, {}^jl_k} \int\limits_{l=-\infty}^{{}^jl_k} \int\limits_{x=l-\frac{b_j}{2}}^{{}^jl_k+\frac{b_j}{2}} \int\limits_{r=x-\frac{b_j}{2}}^{\infty} \hat{f}_j^k(x)\, {}^{j-1}f_{LR}^k(l,r)\, dr\, dx\, dl\end{aligned} \tag{13}$$

where ${}^jC_L^k$ is a normalizing constant. This may be modified to yield

$$\begin{aligned}{}^jf_L^k({}^jl_k) = &\frac{1}{{}^jC_L^k} \\ &\Bigg[\hat{f}_j^k\left({}^jl_k + \frac{b_j}{2}\right) \int\limits_{l=-\infty}^{{}^jl_k} \int\limits_{r={}^jl_k}^{\infty} {}^{j-1}f_{LR}^k(l,r)\, dr\, dl \\ &+ \int\limits_{x={}^jl_k-\frac{b_j}{2}}^{{}^jl_k+\frac{b_j}{2}} \int\limits_{r=x-\frac{b_j}{2}}^{\infty} \hat{f}_j^k(x)\, {}^{j-1}f_{LR}^k({}^jl_k, r)\, dr\, dx\Bigg] \;.\end{aligned} \tag{14}$$

LEMMA. The double integral over the joint density ${}^{j-1}f_{LR}^k(l,r)$ of the form

$$\int\limits_{l=-\infty}^{z} \int\limits_{r=z}^{\infty} {}^{j-1}f_{LR}^k(l,r)\, dr\, dl \tag{15}$$

may be expressed in terms of the marginal densities as

$$\int\limits_{y=-\infty}^{z} \left\{{}^{j-1}f_L^k(y) - {}^{j-1}f_R^k(y)\right\} dy \;. \tag{16}$$

PROOF. The left hand side of (15) may be rewritten as

$$\int\limits_{l=-\infty}^{z} \int\limits_{r=-\infty}^{\infty} {}^{j-1}f_{LR}^k(l,r)\, dr\, dl - \int\limits_{l=-\infty}^{z} \int\limits_{r=-\infty}^{z} {}^{j-1}f_{LR}^k(l,r)\, dr\, dl \;. \tag{17}$$

Using the fact that ${}^{j-1}f_{LR}^{k}(l,r)$ is equal to zero for $l > r$, we may replace the upper limit of the first integral in the second expression by ∞. Interchanging the order of integration in the second expression yields

$$\int_{l=-\infty}^{z} {}^{j-1}f_L^k(l)\,dl - \int_{r=-\infty}^{z} {}^{j-1}f_R^k(r)\,dr. \tag{18}$$

This concludes the proof. □

Eq. (14) may be further simplified by using the lemma and by again using the fact that ${}^{j-1}f_{LR}^{k}(l,r)$ is equal to zero for $l > r$. The lower limit $x - \frac{b_j}{2}$ in the second integral of the second expression may be replaced by $-\infty$.

The desired recursion for the marginal density of the lower bound is now obtained as

$$\begin{aligned} &{}^{j}f_L^k(l) = \\ &\left[\hat{f}_j^k\left(l+\frac{b_j}{2}\right)\int_{y=-\infty}^{l}\left\{{}^{j-1}f_L^k(y) - {}^{j-1}f_R^k(y)\right\}dy \right.\\ &\left. + {}^{j-1}f_L^k(l)\int_{x=l-\frac{b_j}{2}}^{l+\frac{b_j}{2}}\hat{f}_j^k(x)\,dx\right] \Big/ {}^{j}C_L^k\,. \end{aligned} \tag{19}$$

In analogy, the recursion for the marginal density of the upper bound may be derived as

$$\begin{aligned} &{}^{j}f_R^k(r) = \\ &\left[\hat{f}_j^k\left(r-\frac{b_j}{2}\right)\int_{y=-\infty}^{r}\left\{{}^{j-1}f_L^k(y) - {}^{j-1}f_R^k(y)\right\}dy \right.\\ &\left. + {}^{j-1}f_R^k(r)\int_{x=r-\frac{b_j}{2}}^{r+\frac{b_j}{2}}\hat{f}_j^k(x)\,dx\right] \Big/ {}^{j}C_R^k\,, \end{aligned} \tag{20}$$

where ${}^{j}C_R^k$ is a normalizing constant.

This lattice–type recursion for the marginal densities of lower and upper bound is depicted in Fig. 2 and initialized with

$${}^{1}f_L^k = \hat{f}_1^k\left(l+\frac{b_1}{2}\right)\,, \tag{21}$$

$${}^{1}f_R^k = \hat{f}_1^k\left(r-\frac{b_1}{2}\right)\,. \tag{22}$$

4.2 Gaussian Noise Densities

Again, $\hat{f}_j^k$ is a Gaussian density. The above expressions for the marginal densities may be reduced to

$$\begin{aligned} &{}^{j}f_L^k(l) = \frac{1}{{}^{j}C_L^k} \\ &\left[\hat{f}_j^k\left(l+\frac{b_j}{2}\right)\int_{y=-\infty}^{l}\left\{{}^{j-1}f_L^k(y) - {}^{j-1}f_R^k(y)\right\}dy \right.\\ &+ {}^{j-1}f_L^k(l) \\ &\left.\left(\operatorname{erf}\left(\frac{l+\frac{b_j}{2}-\hat{m}_j^k}{\hat{\sigma}_j^k}\right) - \operatorname{erf}\left(\frac{l-\frac{b_j}{2}-\hat{m}_j^k}{\hat{\sigma}_j^k}\right)\right)\right]\,, \end{aligned} \tag{23}$$

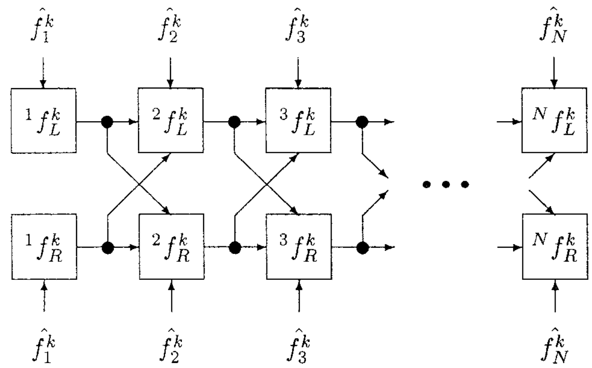

Figure 2: Lattice–type recursion for the marginal densities of lower and upper bound in the case of N information sources.

$$\begin{aligned} &{}^{j}f_R^k(r) = \frac{1}{{}^{j}C_R^k} \\ &\left[\hat{f}_j^k\left(r-\frac{b_j}{2}\right)\int_{y=-\infty}^{r}\left\{{}^{j-1}f_L^k(y) - {}^{j-1}f_R^k(y)\right\}dy \right.\\ &+ {}^{j-1}f_R^k(r) \\ &\left.\left(\operatorname{erf}\left(\frac{r+\frac{b_j}{2}-\hat{m}_j^k}{\hat{\sigma}_j^k}\right) - \operatorname{erf}\left(\frac{r-\frac{b_j}{2}-\hat{m}_j^k}{\hat{\sigma}_j^k}\right)\right)\right]\,. \end{aligned} \tag{24}$$

5 Simulative Verification

Since the authors' background is in mobile robot localization based on optical range data [6], [7], acoustical range data [3], and angular measurements [4], a simple scalar mobile robot localization problem is considered.

Two border cases of gathering information from several sources are illustrated by numerical examples: The first case consists of sampling the first source several times, then sampling the second source, and so on. The second case assumes that samples from all sources are available simultaneously.

Consider a vehicle equipped with a range sensor

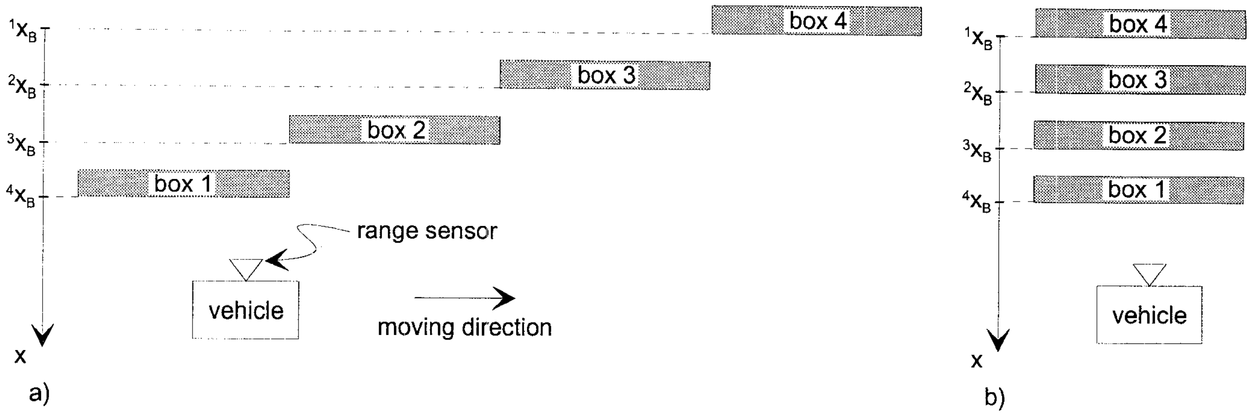

Figure 3: Experimental setup for mobile robot localization: a) sequential sampling b) simultaneous sampling.

Box	1	2	3	4
True value ${}^i\tilde{x}_B$	125	82	28	6
Nominal value ${}^ix_B$	120	80	40	0
Bound b_i	40	20	30	20
Standard deviation σ_i	10	10	10	10

Table 1: Parameters of localization experiment.

that measures the distance to a number of boxes, Fig. 3. The box positions are known within a given geometric tolerance, i.e.,

$$ {}^ix_B = {}^i\tilde{x}_B + {}^i\Delta x_B \ , \qquad \text{with } \left|{}^i\Delta x_B\right| \leq \frac{b_i}{2} \ , \tag{25} $$

where ${}^i\tilde{x}_B$ denotes the unknown true value and ${}^i\Delta x_B$ is the unknown but bounded deviation of the nominal value ${}^ix_B$. The range sensor is corrupted by additive white Gaussian noise with zero mean and a variance σ_i^2 which depends on the surface characteristic of the box i. The measurement equation is thus given by

$$ {}^ix_B + D_i^k = x + {}^i\Delta x_B + {}^iE_s \ , \tag{26} $$

where ${}^iE_s \sim N(0, \sigma_i)$, x denotes the vehicle position, and D_i^k is the measured distance. The two simulations are now performed for a true vehicle position $x = 200$ and the parameters given in Tab. 1.

$\hat{f}_i^k(x)$ is a Gaussian density where the mean and variance are recursively estimated by observing source i as [9]

$$ \hat{m}_i^k = \frac{\left(\hat{\sigma}_i^{k-1}\right)^{-2}\hat{m}_i^{k-1} + (\sigma_i)^{-2}\left({}^ix_B + d_i^k\right)}{\left(\hat{\sigma}_i^{k-1}\right)^{-2} + (\sigma_i)^{-2}} \ , \tag{27} $$

$$ \left(\hat{\sigma}_i^k\right)^2 = \left(\left(\hat{\sigma}_i^{k-1}\right)^{-2} + (\sigma_i)^{-2}\right)^{-1} \ , \tag{28} $$

with $\hat{m}_i^0 = 0$, $\left(\hat{\sigma}_i^0\right)^{-1} = 0$.

The first simulation refers to Fig. 3 a) where the vehicle moves along the four boxes. The boxes are sampled sequentially with 100 samples for each box. We start with the first box, and simply obtain the marginal densities as the shifted versions of $\hat{f}_1^k(x)$ according to (21), (22). These marginals serve as the initial densities for the recursion formulae (23) and (24) that are used for including information sources 2, 3, and 4 sequentially. The SSI recursion step from $i-1$ to i is performed whenever the Bayesian update (27), (28) for information source i has been done, i.e., 100 times for boxes 2, 3, and 4, respectively. Fig. 4 a) depicts the response of the expected values of the lower and upper bound of the set estimate. Sampling a specific box reduces the stochastic uncertainty. The initial deterministic uncertainty given by the interval [175, 215] is obviously reduced each time the box is changed: When changing from box 1 to box 2, the resulting intersection set without stochastic noise would be [188, 208], which is eventually approached when sampling box 2 for an infinite number of times. When traversing from box 2 to box 3, only the lower bound is updated, since the noise-free interval intersection would yield [197, 208]. Switching from box 3 to box 4 produces an update for the upper bound only, since the underlying interval intersection would yield [197, 204].

The second simulation refers to Fig. 3 b) where the vehicle samples all four boxes simultaneously at time k. The Bayesian update (27), (28) is performed at every time k for each source i. Subsequently, the SSI recursions (23), (24) are performed, starting with (21), (22), up to information source 4, at every time k. In this experiment, the underlying noise-free intersection set is [197, 204] for all k. This set is approached for an infinite number of measurements.

Fig. 4 evidently shows the realistic quantification of the associated estimation uncertainty which is in sharp

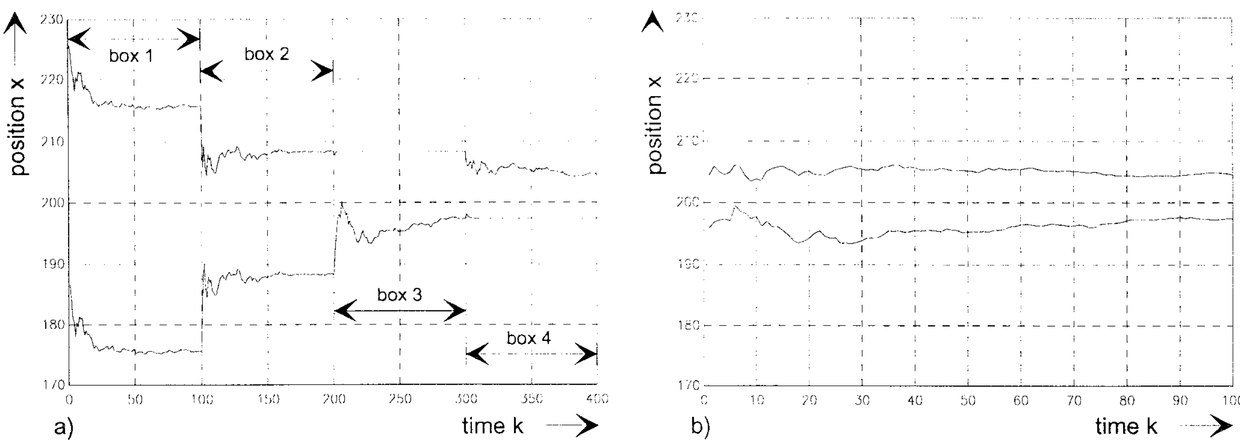

Figure 4: Expected values of lower and upper bound: a) sequential sampling b) simultaneous sampling.

contrast to the optimism of point estimators. This feature may be exploited when attempting to navigate a mobile robot through narrow openings.

6 Conclusions

A combined **S**tatistical and **S**et theoretic **I**nformation (SSI) filter is introduced for fusing the information from several sources which are simultaneously corrupted by a deterministic amplitude–bounded unknown bias error and a possibly unbounded random process. The new approach unites proven schemes for handling pure stochastic noise and for treating amplitude–bounded uncertainties. Set estimates are provided rather than point estimates. Furthermore, the set bounds are uncertain in a statistical sense. Thus, these estimates do not suffer from the over–optimism encountered when just considering one form of uncertainty though both are present. Monte Carlo simulations in the context of mobile robot localization demonstrate the effectiveness of the proposed approach. The simulation results reveal the conceptionally different reductions in uncertainty during the measurement process.

Our study so far only considered the scalar case. Nevertheless, generalization to higher dimensions is straightforward when attention is limited to hyper–rectangles parallel to the coordinate axes. The treatment of the common ellipsoidal set bounds is more involved, since ellipsoids are not closed under intersection, and the detection of ellipsoid overlap is tedious.

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