ELSEVIER

Information Fusion 1 (2000) 55–63

INFORMATION FUSION

www.elsevier.nl/locate/inffus

Fusing information simultaneously corrupted by uncertainties with known bounds and random noise with known distribution

Uwe D. Hanebeck [a,*], Joachim Horn [b]

[a] *Institute of Automatic Control Engineering, Technische Universität München, 80290 München, Germany*
[b] *Siemens AG, Corporate Technology, Information and Communications, 81730 München, Germany*

Received 30 April 1999; received in revised form 15 March 2000; accepted 16 March 2000

Abstract

A new algorithm is derived for estimating the state of a linear dynamic system by fusing uncertain observations, which suffer from two types of uncertainties simultaneously. The first uncertainty is a stochastic process with given distribution. The second uncertainty is only known to be bounded, the exact underlying distribution is unknown. The new fusion algorithm combines set theoretic and stochastic estimation in a rigorous manner and provides a continuous transition between the two classical information fusion concepts. It converges to a set theoretic estimator, when the stochastic error goes to zero, and to a Kalman filter, when the bounded error vanishes. In the mixed noise case, solution sets are provided that are uncertain in a stochastic sense.

Keywords: Data fusion; State estimation; Mixed uncertainty model; Bounded noise; Stochastic noise

1. Introduction

Estimating the state of a dynamic system by fusing uncertain information is a topic of extraordinary importance. In a wide variety of applications, where an appropriate system model together with a stochastic noise model is given, the Kalman filter [1] and its many variations have proven to be usefull [11].

In many cases, however, uncertainties arise, for example from unmodeled dynamics or unmodeled nonlinearities, which cannot satisfactorily be described as stochastic signals with known distribution. In addition, correlated noise terms or systematic errors may be present but neglected for the sake of simplicity. In that case, Kalman filter estimates tend to be overoptimistic [12], i.e., the covariance is underestimated. Several heuristics have been suggested for coping with this problem, which of course do not provide optimal estimators.

In some situations, bounds for these uncertainties can be provided. In that case, set theoretic estimation can be applied [14], which often leads to good results [4]. However, when additional uncorrelated noise is present, the error bounds become unnecessarily conservative.

In [5,8], a concept for information fusion in the presence of both bounded and stochastic uncertainties has been introduced. The proposed algorithm for the case of a scalar state is exact, but computationally complex. In [6,7], an approximate solution for the case of a scalar state has been derived, that is computationally attractive. Furthermore, a generalization towards arbitrary dimensional states and observations of the same dimension has been proposed in [9].

This paper is concerned with updating the estimate of an *arbitrary dimensional state* based on *scalar observations.* For this very relevant case, an approximate solution is derived, that is computationally attractive. Nevertheless, it combines both set theoretic and stochastic estimation in a rigorous manner. It bridges the gap between both estimation schemes, because a set theoretic estimator is obtained, when the stochastic error goes to zero, and a Kalman filter is obtained, when the bounded error vanishes. When both types of uncertainties are present, the new estimator provides solution sets that are uncertain in a stochastic sense. The propagation of estimates suffering from both uncertainties through a dynamic system is discussed in [10].

In Section 2, standard uncertainty models are first reviewed. Then a generalized uncertainty model

* Corresponding author. Tel.: +49-89-289-23412; fax: +49-89-289-28340.
E-mail addresses: uwe.hanebeck@ei.tum.de (U.D. Hanebeck), joachim.horn@mchp.siemens.de (J. Horn).

1566-2535/00/$ - see front matter
PII: S 1566-2535(00)00005-1

combining stochastic and set theoretic models is introduced. In Section 3, standard approaches for information fusion are reviewed. Section 4.1 then presents a new concept for information fusion in case of the generalized uncertainty model. In Section 4.2, the estimation problem is solved on the basis of a sum approximation. An exact second-order description is derived in Section 4.3. In Section 4.4, the results are summarized. In Section 5, a two-dimensional simulation example is presented that further clarifies the conveyed concepts.

1.1. Notation

Vectors are underlined, e.g. $\underline{x}$, matrices are denoted by boldface letters, e.g. $\mathbf{E}$. True values are denoted by a tilde, e.g. $\tilde{\underline{x}}$. Random variables are denoted by capital letters, e.g. $\underline{X}$. Realizations of a random variable $\underline{X}$ are denoted by a small letter $\underline{x}$. $\hat{\underline{x}}$ is used for denoting either the estimate of an unknown true value $\tilde{\underline{x}}$ or for denoting the mean value of a random variable $\underline{X}$. The covariance matrix of a random variable $\underline{X}$ is given by $\mathbf{C}_x$. For simplification, the covariance matrix of a random variable $\underline{X}_s$ is given by $\mathbf{C}_s$. The density of a random variable $\underline{X}$ is written as $f_x(\underline{x})$.

2. Problem formulation

We consider the problem of updating the estimate of a multidimensional state $\tilde{\underline{x}}$ based on a scalar observation $\hat{y}$. The observation process is modeled by a linear measurement equation according to

$$\hat{y} = \underline{H}^{\mathrm{T}}\tilde{\underline{x}} + v_y$$

with scalar observation $\hat{y}$, state vector $\tilde{\underline{x}}$, additive uncertainty v_y, and known vector $\underline{H}$. Furthermore, there exists a prior estimate $\hat{\underline{x}}_p$ of the state vector. $\hat{\underline{x}}_p$ also suffers from an additive uncertainty $\underline{v}_p$ according to

$$\hat{\underline{x}}_p = \tilde{\underline{x}} + \underline{v}_p.$$

2.1. Stochastic uncertainty model

The most popular approach to modeling uncertainty is to adopt a stochastic noise model. Here, we have

$$v_y = c_y, \quad \underline{v}_p = \underline{c}_p,$$

where c_y, $\underline{c}_p$ are assumed to be
- zero mean;
- mutually independent;
- Gaussian distributed

random variables with known covariances according to

$$c_y \sim N(0, C_y), \quad \underline{c}_p \sim \underline{N}(\underline{0}, \mathbf{C}_p).$$

Of course, it is possible to drop the independence assumption and just call for known correlation between c_y and $\underline{c}_p$. Furthermore, it is also possible to drop the Gaussian assumption and assume given moments up to second order.

2.2. Set theoretic uncertainty model

Another common approach is to use a set theoretic uncertainty model

$$v_y = e_y, \quad \underline{v}_p = \underline{e}_p,$$

where we assume no prior information about e_y, $\underline{e}_p$ besides boundedness [14]. The scalar uncertainty e_y is bounded by an interval according to

$$e_y^2 \leqslant E_y.$$

E_y defines the size of the interval. The vector-valued uncertainty $\underline{e}_p$ is typically bounded by an ellipsoidal set according to

$$\underline{e}_p^{\mathrm{T}}\mathbf{E}_p^{-1}\underline{e}_p \leqslant 1.$$

$\mathbf{E}_p$ is a symmetric, positive definite matrix, which defines size and orientation of the ellipsoid.

It is important to note that this uncertainty model is different from assuming a uniformly distributed random variable. Here, the model includes every distribution with the assumed bounds. Typical uncertainties, that naturally fall into this class, are deterministic errors.

2.3. Generalized uncertainty model

The key point of this paper is the use of a generalized uncertainty model unifying stochastic and set theoretic modeling [2,3]. This allows the treatment of systems corrupted by both bounded and stochastic uncertainties simultaneously. Hence, the model is well-suited for, but not limited to, the combination of deterministic/systematic errors and random noise.

Here, we consider the additive combination of the standard uncertainties, i.e., of stochastic and set theoretic uncertainties, according to

$$v_y = e_y + c_y, \quad \underline{v}_p = \underline{e}_p + \underline{c}_p.$$

e_y, $\underline{e}_p$ represent the set theoretic uncertainty part, c_y, $\underline{c}_p$ are random noise terms.

3. Standard information fusion approaches

3.1. Kalman filter

When a stochastic noise model is adopted, a Kalman filter is appropriate for updating the system state estimate according to [1]

$$\hat{\underline{x}}_s = \hat{\underline{x}}_p + \frac{\mathbf{C}_p\underline{H}}{C_y + \underline{H}^{\mathrm{T}}\mathbf{C}_p\underline{H}}(\hat{y} - \underline{H}^{\mathrm{T}}\hat{\underline{x}}_p) \tag{1}$$

with the following recursion for the covariance matrix

$$\mathbf{C}_s = \mathbf{C}_p - \frac{\mathbf{C}_p \underline{H}\,\underline{H}^{\mathrm{T}} \mathbf{C}_p}{C_y + \underline{H}^{\mathrm{T}} \mathbf{C}_p \underline{H}}. \tag{2}$$

3.2. Set theoretic filter

In the case of a bounded uncertainty model, a set theoretic filter is appropriate for updating the system state estimate. The set theoretic filter calculates an ellipsoidal set of possible states, which is consistent with the assumed a priori bounds on e_y and $\underline{e}_p$. The fusion result is then given by the set

$$\chi_s = \left\{ \underline{\xi}_s : (\underline{\xi}_s - \hat{\underline{x}}_s)^{\mathrm{T}} \mathbf{E}_s^{-1} (\underline{\xi}_s - \hat{\underline{x}}_s) \leqslant 1 \right\}$$

with

$$\hat{\underline{x}}_s = \hat{\underline{x}}_p + \lambda \frac{\mathbf{E}_p \underline{H}}{E_y + \lambda \underline{H}^{\mathrm{T}} \mathbf{E}_p \underline{H}} (\hat{y} - \underline{H}^{\mathrm{T}} \hat{\underline{x}}_p)$$

and

$$\mathbf{E}_s = d \mathbf{P}_s$$

with

$$\mathbf{P}_s = \mathbf{E}_p - \lambda \frac{\mathbf{E}_p \underline{H}\,\underline{H}^{\mathrm{T}} \mathbf{E}_p}{E_y + \lambda \underline{H}^{\mathrm{T}} \mathbf{E}_p \underline{H}}$$

and

$$d = 1 + \lambda - \lambda \frac{\left(\hat{y} - \underline{H}^{\mathrm{T}} \hat{\underline{x}}_p\right)^2}{E_y + \lambda \underline{H}^{\mathrm{T}} \mathbf{E}_p \underline{H}}.$$

$\lambda \in [0, \infty)$ is chosen to minimize the size of $\mathbf{E}_s$ [4].

4. Generalized information fusion

4.1. The estimation concept

For deriving an appropriate state estimator, we define

$$\bar{\underline{x}}_p = \hat{\underline{x}}_p - \underline{c}_p, \quad \bar{y} = \hat{y} - c_y.$$

Since there is no prior information about the remaining uncertainties $\underline{e}_p$, e_y besides their boundedness, we make the worst case assumption that $\underline{e}_p$, e_y are fully correlated. Hence, a set theoretic estimator is appropriate for fusing $\bar{y}$ and $\bar{\underline{x}}_p$. The fusion result is then given by the set

$$\chi_s = \left\{ \underline{\xi}_s : (\underline{\xi}_s - \bar{\underline{x}}_s)^{\mathrm{T}} \mathbf{E}_s^{-1} (\underline{\xi}_s - \bar{\underline{x}}_s) \leqslant 1 \right\}$$

with

$$\bar{\underline{x}}_s = \bar{\underline{x}}_p + \lambda \frac{\mathbf{E}_p \underline{H}}{E_y + \lambda \underline{H}^{\mathrm{T}} \mathbf{E}_p \underline{H}} \eta, \quad \eta = \bar{y} - \underline{H}^{\mathrm{T}} \bar{\underline{x}}_p \tag{3}$$

and

$$\mathbf{E}_s = d \mathbf{P}_s, \tag{4}$$

where d is given by

$$d = 1 + \lambda - \lambda \frac{\eta^2}{E_y + \lambda \underline{H}^{\mathrm{T}} \mathbf{E}_p \underline{H}}$$

and $\mathbf{P}_s$ is given by

$$\mathbf{P}_s = \mathbf{E}_p - \lambda \frac{\mathbf{E}_p \underline{H}\,\underline{H}^{\mathrm{T}} \mathbf{E}_p}{E_y + \lambda \underline{H}^{\mathrm{T}} \mathbf{E}_p \underline{H}}.$$

The appropriate selection of the parameter $\lambda \in [0, \infty)$ will be discussed later. (3) can be rewritten as

$$\bar{\underline{x}}_s = \mathbf{W}_x \bar{\underline{x}}_p + \underline{W}_y \bar{y} \tag{5}$$

with

$$\mathbf{W}_x = \mathbf{I} - \lambda \frac{\mathbf{E}_p \underline{H}\,\underline{H}^{\mathrm{T}}}{E_y + \lambda \underline{H}^{\mathrm{T}} \mathbf{E}_p \underline{H}}, \quad \underline{W}_y = \frac{\lambda \mathbf{E}_p \underline{H}}{E_y + \lambda \underline{H}^{\mathrm{T}} \mathbf{E}_p \underline{H}}$$

and

$$\mathbf{W}_x + \underline{W}_y \underline{H}^{\mathrm{T}} = \mathbf{I},$$

where $\mathbf{I}$ denotes the identity matrix.

However, $\bar{\underline{x}}_p$ and $\bar{y}$ cannot be measured directly. Only their noisy counterparts given by $\hat{\underline{x}}_p = \bar{\underline{x}}_p + \underline{c}_p$ and $\hat{y} = \bar{y} + c_y$ are available. Hence, the midpoint of the ellipsoidal set χ_s is a random variable according to

$$\underline{X}_s = \begin{cases} \mathbf{W}_x \underline{X}_p + \underline{W}_y Y & \text{for } |y - \underline{H}^{\mathrm{T}} \underline{x}_p| \leqslant \sqrt{E_y} + \sqrt{\underline{H}^{\mathrm{T}} \mathbf{E}_p \underline{H}}, \\ \text{undefined} & \text{elsewhere,} \end{cases}$$

with random variables $\underline{X}_p$, Y and their realizations $\underline{x}_p, y$, respectively.

To simplify the following derivations, we note that the set theoretic uncertainty $\mathbf{E}_s$ given by (4) depends on $\bar{y}$ and $\bar{\underline{x}}_p$. Setting $\eta = 0$ leads to $d = 1 + \lambda$ and is equivalent to bounding $\mathbf{E}_s$ from above. The resulting $\mathbf{E}_s$ is then given by

$$\mathbf{E}_s = (1 + \lambda) \mathbf{E}_p - (1 + \lambda) \lambda \frac{\mathbf{E}_p \underline{H}\,\underline{H}^{\mathrm{T}} \mathbf{E}_p}{E_y + \lambda \underline{H}^{\mathrm{T}} \mathbf{E}_p \underline{H}}. \tag{6}$$

Since the simplified $\mathbf{E}_s$ in (6) does not depend on the actual values, it is not a random variable.

In this paper, we provide two different solutions for estimating the unknown state vector $\tilde{\underline{x}}$:

- An approximation of the density of $\underline{X}_s$ by a sum (Section 4.2), which in contrast to the exact density can be used for recursive estimation.
- An exact second-order description, i.e., mean and covariance (Section 4.3).

4.2. Calculating the density

Since the random variable $\underline{X}_s$ is defined only for

$$|y - \underline{H}^{\mathrm{T}} \underline{x}_p| \leqslant K \quad \text{with } K = \sqrt{E_y} + \sqrt{\underline{H}^{\mathrm{T}} \mathbf{E}_p \underline{H}},$$

the density of $\underline{X}_s$ is given by [13]

$$f_{\underline{x}_s}(\underline{x}_s) = \frac{1}{|\mathbf{W}_x|} \int_{-\infty}^{\infty} f_{\underline{x}_p y}\left(\mathbf{W}_x^{-1}(\underline{x}_s - \underline{W}_y y), y\right) \mathrm{d}y, \tag{7}$$

where $\mathbf{W}_x$ is assumed to be regular and

$$f_{\underline{x}_p y}(\underline{x}_p, y) = \begin{cases} c_1 f_{\underline{x}_p}(\underline{x}_p) f_y(y) & \text{for } |y - \underline{H}^{\mathrm{T}} \underline{x}_p| \leqslant K, \\ 0 & \text{elsewhere,} \end{cases}$$

with normalizing constant c_1. $f_{\underline{x}_p}(\underline{x}_p)$ and $f_y(y)$ are defined as

$$f_{\underline{x}_p}(\underline{x}_p) = c_2 \exp\left\{ -\frac{1}{2} (\underline{x}_p - \hat{\underline{x}}_p)^{\mathrm{T}} \mathbf{C}_p^{-1} (\underline{x}_p - \hat{\underline{x}}_p) \right\}$$

and

$$f_y(y) = c_3 \exp\left\{ -\frac{1}{2} \frac{(y - \hat{y})^2}{C_y} \right\}$$

with normalizing constants c_2, c_3.

Calculating the exact density $f_{\underline{x}_s}(\underline{x}_s)$ directly from (7) gives a rather complicated and not very useful expression. Hence, a series expansion will be calculated instead. For that purpose, we use

$$\mathrm{rect}(x) = \begin{cases} 1 & \text{for } |x| \leqslant 1, \\ 0 & \text{elsewhere,} \end{cases}$$

to interpret the constraint $|y - \underline{H}^{\mathrm{T}} \underline{x}_p| \leqslant K$ as $\mathrm{rect}((y - \underline{H}^{\mathrm{T}} \underline{x}_p)/K)$, which is then approximated by a weighted Gaussian sum

$$\mathrm{rect}\left(\frac{y - \underline{H}^{\mathrm{T}} \underline{x}_p}{K}\right) \approx \frac{1}{\sqrt{2\pi} c} \sum_{i=-L}^{L} \exp\left\{ -\frac{1}{2} \frac{(y - \underline{H}^{\mathrm{T}} \underline{x}_p - m_g^i)^2}{C_g} \right\} \tag{8}$$

with $m_g^i = i(K/L)$, $C_g = c(K/L)$. Note that the integral from $-\infty$ to ∞ over the sum yields $2K$ independent of L. The free parameter $c \in (0, \infty)$ may be obtained by a one-dimensional search to give the best approximation of the rect-function according to a given norm.

Based on this approximation of the rect-function, the exact density $f_{\underline{x}_s}$ in (7) can be approximated by a sum of simple densities. For that purpose, we first consider one term of the sum (8) which gives

$$f_{\underline{x}_s}^i(\underline{x}_s) = \frac{c_4}{|\mathbf{W}_x|} \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2} \left[(\underline{x}_p - \hat{\underline{x}}_p)^{\mathrm{T}} \mathbf{C}_p^{-1} (\underline{x}_p - \hat{\underline{x}}_p) + \frac{1}{C_y} (y - \hat{y})^2 + \frac{1}{C_g} \left(y - \underline{H}^{\mathrm{T}} \underline{x}_p - m_g^i \right)^2 \right] \right\} \mathrm{d}y$$

for $i = -L, \ldots, L$, $\underline{x}_p = \mathbf{W}_x^{-1}[\underline{x}_s - \underline{W}_y y]$, and normalizing constant c_4. A tedious calculation reveals that this approximation can be simplified to

$$f_{\underline{x}_s}^i(\underline{x}_s) = c_5 g_i \exp\left\{ -\frac{1}{2} \left(\underline{x}_s - \hat{\underline{x}}_s^i \right)^{\mathrm{T}} (\mathbf{C}_s^i)^{-1} \left(\underline{x}_s - \hat{\underline{x}}_s^i \right) \right\}$$

with normalizing constant c_5, weighting factors

$$g_i = \exp\left\{ -\frac{1}{2} \frac{\left(\hat{y} - \underline{H}^{\mathrm{T}} \hat{\underline{x}}_p - m_g^i \right)^2}{\underline{H}^{\mathrm{T}} \mathbf{C}_p \underline{H} + C_y + C_g} \right\},$$

and individual means

$$\hat{\underline{x}}_s^i = \mathbf{W}_x \hat{\underline{x}}_p + \underline{W}_y \hat{y} + \frac{\mathbf{W}_x \mathbf{C}_p \underline{H} - \underline{W}_y C_y}{\underline{H}^{\mathrm{T}} \mathbf{C}_p \underline{H} + C_y + C_g} \left(\hat{y} - \underline{H}^{\mathrm{T}} \hat{\underline{x}}_p - m_g^i \right)$$

for $i = -L, \ldots, L$. The covariance matrices are the same for each term in the sum and given by

$$\mathbf{C}_s^i = \mathbf{W}_x \mathbf{C}_p \mathbf{W}_x^{\mathrm{T}} + \underline{W}_y \underline{W}_y^{\mathrm{T}} C_y - \frac{(\mathbf{W}_x \mathbf{C}_p \underline{H} - \underline{W}_y C_y)(\mathbf{W}_x \mathbf{C}_p \underline{H} - \underline{W}_y C_y)^{\mathrm{T}}}{\underline{H}^{\mathrm{T}} \mathbf{C}_p \underline{H} + C_y + C_g}.$$

The approximate solution for the density $f_{\underline{x}_s}$ is then given by

$$f_{\underline{x}_s}(\underline{x}_s) \approx \sum_{i=-L}^{L} f_{\underline{x}_s}^i(\underline{x}_s), \tag{9}$$

which is a weighted sum of Gaussian densities, where the weighting factors g_i are themselves values of a Gausssian function.

Note: It can be proven that this approximation converges to the exact density for $L \to \infty$.

4.3. Exact analytic solutions for mean and covariance

In the following, an *exact* second-order description for $\underline{X}_s$, i.e., mean $\hat{\underline{x}}_s$ and covariance $\mathbf{C}_s$, will be derived.

4.3.1. Exact analytic solution for the mean

An approximate expression for the mean or expected value $\hat{\underline{x}}_s = E[\underline{X}_s]$ of $\underline{X}_s$ is given by

$$\hat{\underline{x}}_s \approx \frac{\sum_{i=-L}^{L} g_i \hat{\underline{x}}_s^i}{\sum_{i=-L}^{L} g_i}.$$

For $L \to \infty$, this expression gives the exact mean $\hat{\underline{x}}_s$. $L \to \infty$ also implies $C_g \to 0$, and the summation can be replaced by integration. A lengthy calculation gives

$$\hat{\underline{x}}_s = \mathbf{W}_x \hat{\underline{x}}_p + \underline{W}_y \hat{y} - F_1\left(\hat{y} - \underline{H}^{\mathrm{T}} \hat{\underline{x}}_p \right) (\mathbf{W}_x \mathbf{C}_p \underline{H} - \underline{W}_y C_y) \tag{10}$$

with $F_1\left(\hat{y} - \underline{H}^{\mathrm{T}} \hat{\underline{x}}_p \right)$ according to Appendix A.

4.3.2. Exact solution for the covariance

For obtaining the exact covariance of $\underline{X}_s$, the first step is to calculate the covariance of the (approximate) Gaussian sum density $f_{\underline{x}_s}(\underline{x}_s)$ in (9) based on the relation

$$\mathbf{C}_s = \frac{\sum_{i=-L}^{L} g_i \left\{ \mathbf{C}_s^i + \hat{\underline{x}}_s^i (\hat{\underline{x}}_s^i)^{\mathrm{T}} \right\}}{\sum_{i=-L}^{L} g_i} - \hat{\underline{x}}_s \hat{\underline{x}}_s^{\mathrm{T}},$$

which gives the exact $\mathbf{C}_s$ for $L \to \infty$ as

$$\mathbf{C}_s = \mathbf{W}_x \mathbf{C}_p \mathbf{W}_x^{\mathrm{T}} + \underline{W}_y \underline{W}_y^{\mathrm{T}} C_y - F_2\left(\hat{y} - \underline{H}^{\mathrm{T}} \hat{\underline{x}}_p\right) \left(\mathbf{W}_x \mathbf{C}_p \underline{H} - \underline{W}_y C_y\right) \left(\mathbf{W}_x \mathbf{C}_p \underline{H} - \underline{W}_y C_y\right)^{\mathrm{T}} \quad (11)$$

with $F_2(\hat{y} - \underline{H}^{\mathrm{T}} \hat{\underline{x}}_p)$ according to Appendix A.

4.4. The new estimator

In Section 4.2, it has been shown that the uncertainty of the fusion result is given by a bounded uncertainty and a sum of Gaussian densities. When the number of terms included in the Gaussian sum tends towards infinity, the exact density is approached. In addition, mean and covariance of the exact density have been given in closed form. These important results can now be applied to derive a second-order estimator for solving practical estimation problems.

This estimator keeps second-order information on both the set theoretic and the stochastic uncertainty. The set theoretic uncertainty is given by the ellipsoidal set derived in Section 4.1. The stochastic uncertainty is given by the exact mean and covariance derived in Section 4.3. The parameter λ is chosen such that an appropriate measure of the total uncertainty is minimized. Here, minimization of $\det(\mathbf{E}_s) + \det(\mathbf{C}_s)$ has been performed numerically. Closed-form solutions for optimal values of λ for different uncertainty measures are available, but outside the scope of this paper.

The estimator unifies Kalman filtering and set theoretic estimation: A Kalman filter is approached, when the bounded error vanishes. On the other hand, a set theoretic estimator is attained, when the stochastic error goes to zero. When both types of uncertainties are present simultaneously, the new estimator provides solution sets that are uncertain in a stochastic sense (see Fig. 1).

5. Simulation example

Consider a vehicle equipped with range sensors that measure the distances to two walls i, $i = 1, 2$, Fig. 2. The wall positions are known within a given geometric tolerance, i.e.,

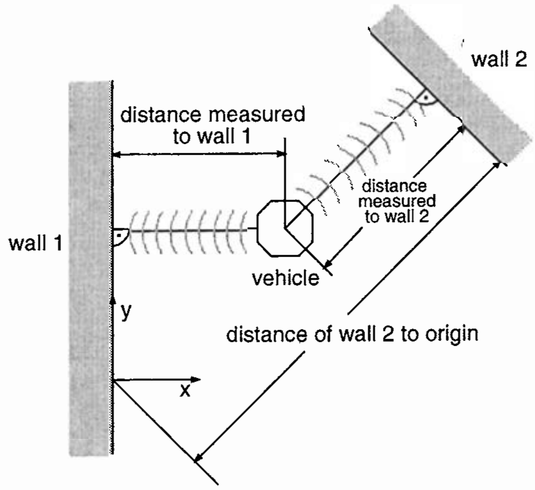

Fig. 2. Setup for simulative example.

$$\hat{d}_i = \tilde{d}_i + \Delta d_i \quad \text{with } |\Delta d_i| \leqslant b_i,$$

where $\tilde{d}_i$ denotes the unknown true (signed) distance of the wall to the origin and Δd_i is the unknown but bounded deviation of the nominal value $\hat{d}_i$. The corresponding unit normal vector $\underline{H}_i$ is assumed to be known. The range measurements are corrupted by additive white Gaussian noise with zero mean and a variance σ_i^2. The measurement equation is given by

$$\hat{y}^k = \hat{d}_i + \hat{D}_i^k = \underline{H}_i^{\mathrm{T}} \tilde{\underline{x}} + \Delta d_i + c_i^k,$$

where $c_i^k \sim N(0, \sigma_i)$, $\tilde{\underline{x}}$ denotes the vehicle position, and $\hat{D}_i^k$ is the measured distance. A true vehicle position $\tilde{\underline{x}} = [2000, 2000]^{\mathrm{T}}$ is assumed. The remaining parameters are given in Table 1. The initial position estimate is given by $\hat{\underline{x}}_s^0 = [1900, 2100]^{\mathrm{T}}$ with $\mathbf{E}_s^0 = \mathrm{diag}(2000^2, 2000^2)$ and $\mathbf{C}_s^0 = \mathrm{diag}(2000^2, 2000^2)$. At each time step k, the distances to both walls are measured.

Fusion with the Kalman filter: The Kalman filter is evaluated by recursively updating the position estimate using the equation (1) for $\hat{\underline{x}}_s^k$ and (2) for $\mathbf{C}_s^k$. To employ standard Kalman filtering, the wall uncertainties are viewed as additional uncorrelated noise terms. For wall $i = 1, 2$, we obtain a total measurement variance of $C_y^k + E_y^k = \sigma_i^2 + b_i^2$. At every time step k, the filter is applied twice: The measured distance to wall 1 yields an

$$\hat{\underline{x}}_s = \mathbf{W}_x \hat{\underline{x}}_p + \underline{W}_y \hat{y} - F_1\left(\hat{y} - \underline{H}^T \hat{\underline{x}}_p\right) \left(\mathbf{W}_x \mathbf{C}_p \underline{H} - \underline{W}_y C_y\right)$$

$$\mathbf{E}_s = (1+\lambda)\,\mathbf{E}_p - (1+\lambda)\,\lambda \frac{\mathbf{E}_p \underline{H}\,\underline{H}^T \mathbf{E}_p}{E_y + \lambda \underline{H}^T \mathbf{E}_p \underline{H}}$$

$$\mathbf{C}_s = \mathbf{W}_x \mathbf{C}_p \mathbf{W}_x^T + \underline{W}_y \underline{W}_y^T C_y - F_2\left(\hat{y} - \underline{H}^T \hat{\underline{x}}_p\right) \left(\mathbf{W}_x \mathbf{C}_p \underline{H} - \underline{W}_y C_y\right) \left(\mathbf{W}_x \mathbf{C}_p \underline{H} - \underline{W}_y C_y\right)^T$$

$$\mathbf{W}_x = \mathbf{I} - \lambda \frac{\mathbf{E}_p \underline{H}\,\underline{H}^T}{E_y + \lambda \underline{H}^T \mathbf{E}_p \underline{H}}, \quad \underline{W}_y = \frac{\lambda\,\mathbf{E}_p \underline{H}}{E_y + \lambda \underline{H}^T \mathbf{E}_p \underline{H}}$$

Fig. 1. Summary of equations for the new estimator: Mean value $\hat{\underline{x}}_s$, set theoretic uncertainty $\mathbf{E}_s$, stochastic uncertainty $\mathbf{C}_s$, and weighting factors $\mathbf{W}_x$, $\underline{W}_y$. $F_1(\hat{y} - \underline{H}^{\mathrm{T}} \hat{\underline{x}}_p)$ and $F_2(\hat{y} - \underline{H}^{\mathrm{T}} \hat{\underline{x}}_p)$ are given in Appendix A.

Table 1
Parameters of localization experiment

Wall	1	2
Unit normal vector, $\underline{H}_i$	$[1,0]^T$	$-1/\sqrt{2}[1,1]^T$
Nominal distance, $\hat{d}_i$	0	−6000
True distance, $\tilde{d}_i$	−40	−6030
Bound, b_i	50	50
Standard deviation, σ_i	100	100

intermediate estimate, and the measured distance to wall 2 yields the estimate $\hat{\underline{x}}_s^k$ that incorporates all measurements available up to time k. The evolution of the resulting confidence set is depicted in Fig. 3 for $k = 1, 2, 3, 10, 100, 1000$. The optimal estimate for an infinite number of measurements would be the set resulting from intersecting the two strips that correspond to the uncertainty of the two walls. The true state $\tilde{\underline{x}} = [2000, 2000]^T$ is marked by a dot. The confidence set has been calculated based on nine times the Kalman filter covariance matrix $\mathbf{C}_s^k$ centered at $\hat{\underline{x}}_s^k$.

Note: The confidence set for $k \to \infty$ does *not* contain the true state.

Fusion with the set theoretic estimator: The set theoretic estimator is applied in a similar fashion. Here, the noise in the distance measurements is included by increasing the measurement uncertainty according to $E_y^k + \gamma C_y^k = b_i^2 + \gamma\sigma_i^2$, $i = 1, 2$. To consider the long tails of the assumed Gaussian distribution, γ must be chosen very conservative. The smaller the choice of γ, the higher the probability of producing a wrong estimate.

The result for $\gamma = 4$ is shown in Fig. 4.

Note: The confidence set for $k \to \infty$ does *not* contain the true state. Application of $\gamma = 9$ is shown in Fig. 5. Here, the estimator is consistent, but gives a very conservative result.

Fusion with the new estimator: The proposed new estimator is evaluated by recursively updating the position estimate using the equations for $\hat{\underline{x}}_s^k$ in (10), $\mathbf{E}_s^k$ in (6), and $\mathbf{C}_s^k$ in (11) twice: Once for wall 1 with $E_y^k = b_1^2$ and $C_y^k = \sigma_1^2$, which yields an intermediate estimate, and once for wall 2 with $E_y^k = b_2^2$ and $C_y^k = \sigma_2^2$, which yields the estimate $\hat{\underline{x}}_s^k$ that incorporates all measurements available up to time k. The parameter λ_k is chosen such that $\det(\mathbf{E}_s^k) + \det(\mathbf{C}_s^k)$ is minimized. Fig. 6 depicts how the resulting estimate evolves over time. Here, the confidence set is given as the Minkowski sum of $\mathbf{E}_s^k$ and $9\,\mathbf{C}_s^k$ centered at $\hat{\underline{x}}_s^k$.

Note: The confidence set for $k \to \infty$ bounds the exact set (the intersection of the two strips) from above and hence contains the true state.

6. Conclusions

Many problems in information fusion can be attacked in a mixed noise setting, i.e., the arising uncertainties can be modeled as being additively composed of both: (1) noise with known bounds and (2) noise with known distribution. For these problems, a new fusion algorithm has been derived for the important case of an arbitrary dimensional state and scalar measurements,

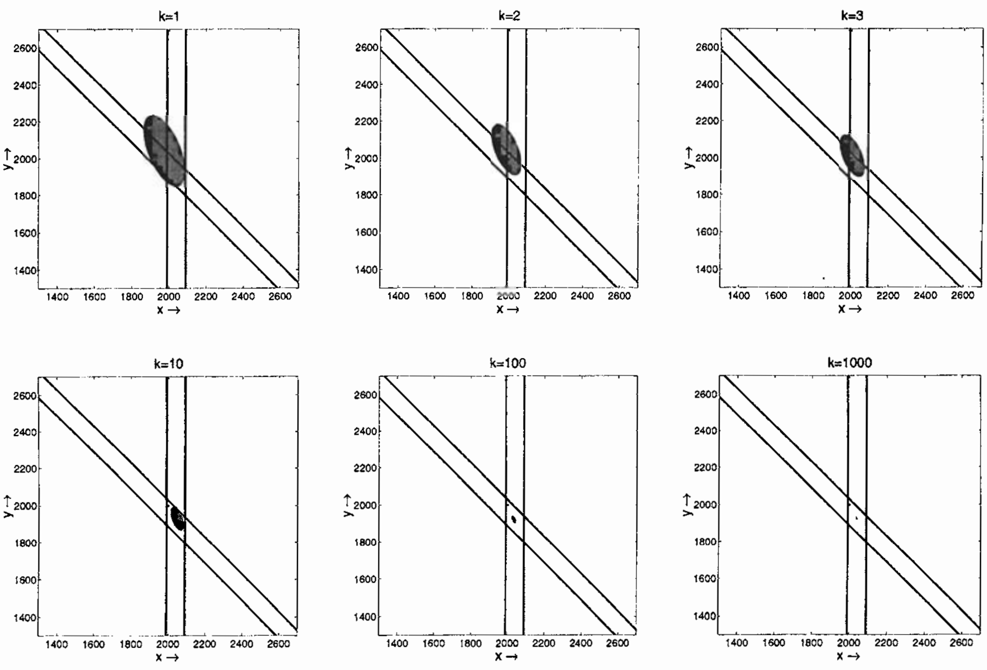

Fig. 3. Results of fusion with the Kalman filter: Evolution of confidence sets over time.

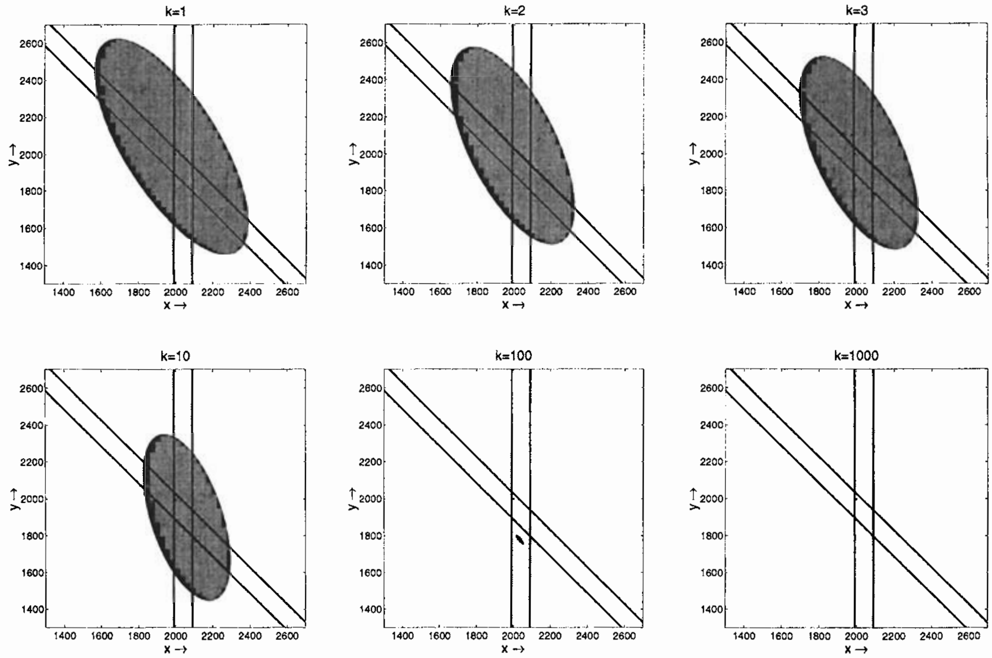

Fig. 4. Results of fusion with the set theoretic filter ($\gamma = 4$): Evolution of confidence sets over time.

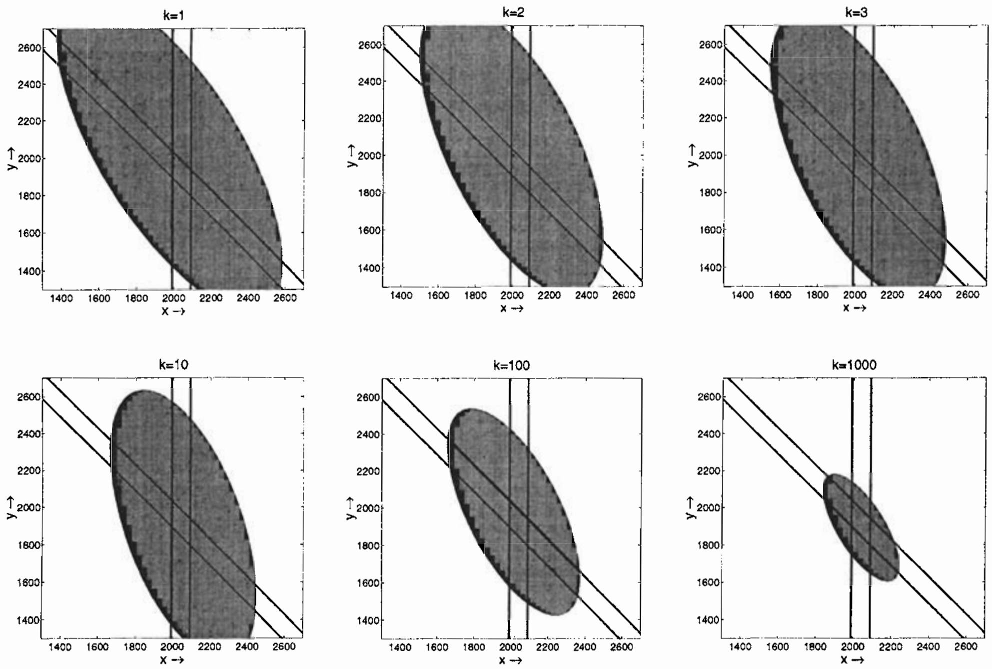

Fig. 5. Results of fusion with the set theoretic filter ($\gamma = 9$): Evolution of confidence sets over time.

which combines set theoretic and stochastic fusion in a rigorous manner. Hence, it provides solution sets that are uncertain in a stochastic sense. The proposed fusion algorithm is efficient and, hence, well-suited for practical applications.

This paper focused on the measurement update, i.e., on updating the estimate of an arbitrary dimensional state based on given scalar observations. The time update, i.e., propagating the state estimate through a dynamic system model, is discussed in [10].

Appendix A

The nonlinear functions $F_1(\hat{y} - \underline{H}^{\mathrm{T}}\hat{\underline{x}}_p)$, $F_2(\hat{y} - \underline{H}^{\mathrm{T}}\hat{\underline{x}}_p)$ of the innovation $\hat{y} - \underline{H}^{\mathrm{T}}\hat{\underline{x}}_p$ are given by

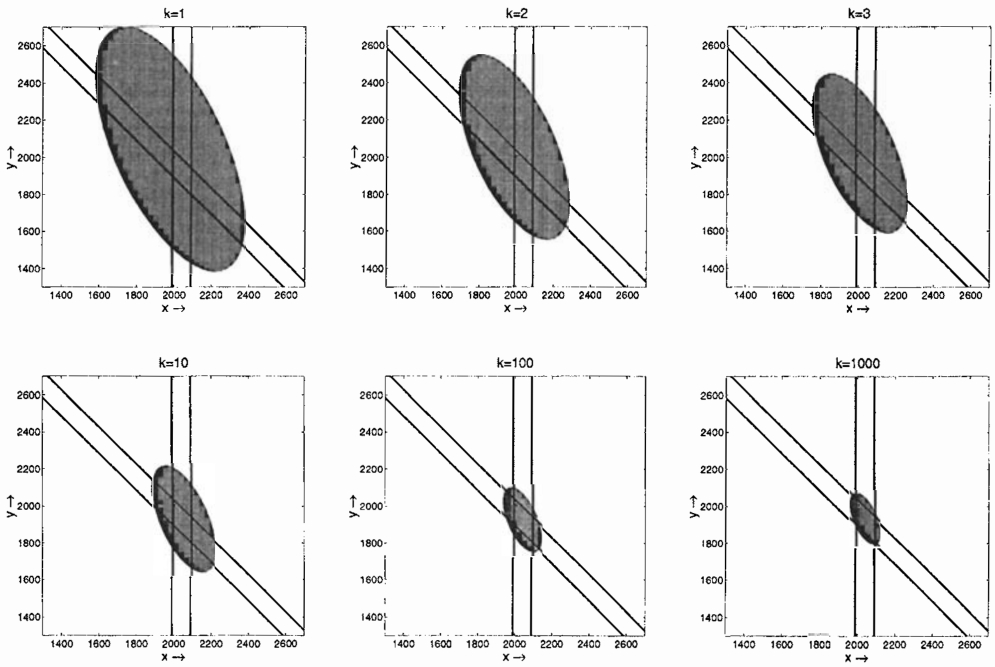

Fig. 6. Results of fusion with the new estimator: Evolution of confidence sets over time.

$$F_1\left(\hat{y}-\underline{H}^{\mathrm{T}}\hat{\underline{x}}_p\right) = G_0\left(\hat{y}-\underline{H}^{\mathrm{T}}\hat{\underline{x}}_p, \sqrt{E_y}+\sqrt{\underline{H}^{\mathrm{T}}\mathbf{E}_p\underline{H}}, C_y+\sqrt{\underline{H}^{\mathrm{T}}\mathbf{C}_p\underline{H}}\right),$$

$$F_2\left(\hat{y}-\underline{H}^{\mathrm{T}}\hat{\underline{x}}_p\right) = \left[G_0\left(\hat{y}-\underline{H}^{\mathrm{T}}\hat{\underline{x}}_p, \sqrt{E_y}+\sqrt{\underline{H}^{\mathrm{T}}\mathbf{E}_p\underline{H}}, C_y+\sqrt{\underline{H}^{\mathrm{T}}\mathbf{C}_p\underline{H}}\right)\right]^2 + \frac{G_1\left(\hat{y}-\underline{H}^{\mathrm{T}}\hat{\underline{x}}_p, \sqrt{E_y}+\sqrt{\underline{H}^{\mathrm{T}}\mathbf{E}_p\underline{H}}, C_y+\sqrt{\underline{H}^{\mathrm{T}}\mathbf{C}_p\underline{H}}\right)}{C_y+\underline{H}^{\mathrm{T}}\mathbf{C}_p\underline{H}}$$

with functions G_0 and G_1

$$G_0(x,B,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\left[\left(\exp\left\{-\frac{1}{2}\frac{(x-B)^2}{\sigma^2}\right\} - \exp\left\{-\frac{1}{2}\frac{(x+B)^2}{\sigma^2}\right\}\right)\Big/\left(\operatorname{erf}\left\{\frac{x-B}{\sigma}\right\} - \operatorname{erf}\left\{\frac{x+B}{\sigma}\right\}\right)\right],$$

$$G_1(x,B,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\left[\left((x-B)\exp\left\{-\frac{1}{2}\frac{(x-B)^2}{\sigma^2}\right\} - (x+B)\exp\left\{-\frac{1}{2}\frac{(x+B)^2}{\sigma^2}\right\}\right)\Big/\left(\operatorname{erf}\left\{\frac{x-B}{\sigma}\right\} - \operatorname{erf}\left\{\frac{x+B}{\sigma}\right\}\right)\right]$$

and $\operatorname{erf}(x)$ defined according to [13].

References

[1] B.D.O. Anderson, J.B. Moore, Optimal Filtering, Prentice-Hall, Englewood Cliffs, NJ, 1979.

[2] V.Z. Filipovic, Identification in H_∞ in the presence of deterministic and stochastic noise, in: Proceedings of the Third European Control Conference (ECC '95), Rome, Italy, September 1995, pp. 31–35.

[3] K. Forsman, L. Ljung, On the dead-zone in system identification, in: C. Banyasz, L. Keviczky (Eds.), Identification and System Parameter Estimation 1991, Selected Papers from the Ninth IFAC/IFORS Symposium on System Identification, Budapest, Hungary, July 1991, Pergamon, Oxford, UK, 1992, pp. 959–963.

[4] U.D. Hanebeck, G. Schmidt, Set-theoretic localization of fast mobile robots using an angle measurement technique, in: Proceedings of the 1996 IEEE International Conference on Robotics and Automation (ICRA '96), Minneapolis, Minnesota, USA, April 1996, pp. 1387–1394.

[5] U.D. Hanebeck, J. Horn, G. Schmidt, On combining set theoretic and bayesian estimation, in: Proceedings of the 1996 IEEE International Conference on Robotics and Automation (ICRA '96), Minneapolis, Minnesota, USA, April 1996, pp. 3081–3086.

[6] U.D. Hanebeck, J. Horn, A new state estimator for a mixed stochastic and set theoretic uncertainty model, in: Proceedings of SPIE vol. 3720, AeroSense Symposium, Orlando, Florida, USA, April 1999, pp. 336–344.

[7] U.D. Hanebeck, J. Horn, A new estimator for mixed stochastic and set theoretic uncertainty models applied to mobile robot localization, in: Proceedings of the 1999 IEEE International Conference on Robotics and Automation (ICRA '99), Detroit, Michigan, USA, May 1999, pp. 1335–1340.

[8] U.D. Hanebeck, J. Horn, G. Schmidt, On combining statistical and set-theoretic estimation, Automatica 35 (6) (1999), pp. 1101–1109.

[9] U.D. Hanebeck, J. Horn, New estimators for mixed stochastic and set theoretic uncertainty models: The vector case, in: Proceedings

of the Fifth European Control Conference (ECC '99), Karlsruhe, Germany, August/September, 1999.

[10] U.D. Hanebeck, J. Horn, New results for state estimation in the presence of mixed stochastic and set theoretic uncertainties, in: Proceedings of the 1999 IEEE Systems, Man, and Cybernetics Conference (SMC '99), Tokyo, Japan, October 1999, vol. 5, pp. 86–91.

[11] J. Horn, G. Schmidt, Continuous localization for long-range indoor navigation of mobile robots, in: Proceedings of the 1995 IEEE International Conference on Robotics and Automation (ICRA '95), Nagoya, Japan, May 1995, pp. 387–394.

[12] S.J. Julier, J.K. Uhlmann, A non-divergent estimation algorithm in the presence of unknown correlations, in: Proceedings of the 1997 American Control Conference (ACC '97), Albuquerque, New Mexico, USA, June 1997, pp. 2369–2373.

[13] A. Papoulis, Probability, Random Variables and Stochastic Processes, Second Edition, McGraw-Hill, New York, 1984.

[14] F.C. Schweppe, Uncertain Dynamic Systems, Prentice-Hall, Englewood Cliffs, NJ, 1973.