State Estimation Based on Observations Simultaneously Corrupted by Random Noise with Known Distribution and Uncertainties with Known Bounds

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ABSTRACT

This paper presents a new approach for estimating the state of a linear dynamic system when two different types of uncertainties are present simultaneously. The first type of uncertainty is a stochastic process with given distribution. The second type of uncertainty is only known to be bounded, the exact underlying distribution is unknown. This includes inequality constraints between state variables, geometric tolerances, and bounded noise sources which are possibly correlated. For this generalized uncertainty model, a new recursive estimator has been developed. The new estimator unifies Kalman filtering and set theoretic filtering. It converges to a Kalman filter, when the bounded uncertainty goes to zero, and it converges to a set theoretic filter, when the stochastic noise vanishes. In the case of mixed uncertainties, the new estimator provides solution sets that are uncertain in a statistical sense.

1 Introduction

We consider the problem of determining a state $\underline{x}$ based on scalar measurements y_k, which are corrupted by *additive* uncertainties v_k according to

$$y_k = \underline{H}_k^T \underline{x} + v_k \ .$$

These uncertainties can, for example, be described in a stochastic setting. A stochastic model is, for example, appropriate for describing thermal noise. In that case, a Kalman filter can be used for estimating the system state [13].

On the other hand, the uncertainties can be modeled as being bounded with no underlying distribution assumed. This is useful for including inequality constraints between state variables, geometric tolerances, and bounded noise sources which are possibly correlated. For the case of bounded uncertainties, a set theoretic filter is the appropriate tool for estimating the system state [15].

Many real–world problems can be described by a combination of the two types of uncertainties, i.e., of stochastic and set theoretic uncertainties. This is essential when including noisy constraints or when considering the additive combination of noise with known distribution and noise with known bounds.

In [5, 8], a concept for state estimation in the presence of both set theoretic and stochastic uncertainties has been introduced. The proposed algorithm for the case of a scalar state is exact, but computationally complex. In [6, 7], an approximate solution for the case of a scalar state has been derived, that is computationally attractive. Furthermore, a generalization towards arbitrary dimensional states and observations of the same dimension has been proposed in [9].

This paper is concerned with estimating arbitrary dimensional states $\underline{x}$ based on scalar observations y_k. For this very relevant case, a new, approximate solution for the measurement update is derived, that is computationally attractive. Nevertheless, it combines both stochastic and set theoretic estimation in a rigorous manner. It bridges the gap between both estimation schemes, because a Kalman filter is attained, when the bounded error goes to zero, and a set theoretic estimator is attained, when the stochastic error vanishes. When both types of uncertainty are present, the new estimator provides solution sets that are uncertain in a statistical sense.

The propagation of these estimates through a dynamic system, the so–called time update step, is given in [10].

In Section 2, the Kalman filter for recursively estimating the system state in the presence of stochastic uncertainties is reviewed for the case of a linear system and scalar observations. Section 3 presents a review of the set theoretic filter for state estimation with bounded uncertainties also for the case of a linear system and scalar observations. The new filter is then introduced in Section 4. In Section 5, a two–dimensional simulative example for comparing the different filter concepts is presented.

2 Kalman Filter

When a stochastic noise model is adopted, a Kalman filter is appropriate for estimating the system state. Here, we have $v_k = c_k$, where c_k is assumed to be a zero mean, white, Gaussian random process with known variance according to $c_k \sim N(0,\ C_k^y)$. Of course, the whiteness assumption could in general be replaced by assuming colored noise produced by a known dynamic system driven by white noise [1]. Furthermore, it is also possible to drop the Gaussian assumption and assume given moments up to second order only.

The observation y_k at time k is used to perform the measurement update according to

$$\underline{x}_k = \underline{x}_{k-1} + \frac{\mathbf{C}_{k-1}\underline{H}_k}{C_k^y + \underline{H}_k^T \mathbf{C}_{k-1}\underline{H}_k}(y_k - \underline{H}_k^T \underline{x}_{k-1}) \ , \tag{1}$$

with the following recursion for the covariance matrix

$$\mathbf{C}_k = \mathbf{C}_{k-1} - \frac{\mathbf{C}_{k-1}\underline{H}_k\,\underline{H}_k^T\mathbf{C}_{k-1}}{C_k^y + \underline{H}_k^T \mathbf{C}_{k-1}\underline{H}_k} \ . \tag{2}$$

3 Set Theoretic Filter

In the case of a bounded uncertainty model, a set theoretic filter is appropriate for estimating the system state. Here, we have $v_k = e_k$ where we assume no prior information about e_k besides boundedness according to $e_k^2 \leq E_k^y$. Essentially, this means that the underlying distribution of e_k is unknown. Hence, this class of uncertainties includes systematic, correlated, and fully dependent errors.

The observation y_k at time k is used to perform the measurement update according to

$$\underline{x}_k = \underline{x}_{k-1} + \lambda_k \frac{\mathbf{E}_{k-1}\underline{H}_k}{E_k^y + \lambda_k\,\underline{H}_k^T \mathbf{E}_{k-1}\underline{H}_k}\left(y_k - \underline{H}_k^T \underline{x}_{k-1}\right)$$

and

$$\mathbf{E}_k = d_k \mathbf{P}_k$$

with

$$\mathbf{P}_k = \mathbf{E}_{k-1} - \lambda_k \frac{\mathbf{E}_{k-1}\underline{H}_k\,\underline{H}_k^T\mathbf{E}_{k-1}}{E_k^y + \lambda_k\,\underline{H}_k^T \mathbf{E}_{k-1}\underline{H}_k}$$

and

$$d_k = 1 + \lambda_k - \frac{\lambda_k\left(y_k - \underline{H}_k^T \underline{x}_{k-1}\right)^2}{E_k^y + \lambda_k\,\underline{H}_k^T \mathbf{E}_{k-1}\underline{H}_k} \ .$$

$\lambda_k \in [0, \infty)$ is chosen to minimize the size of $\mathbf{E}_k$ [4].

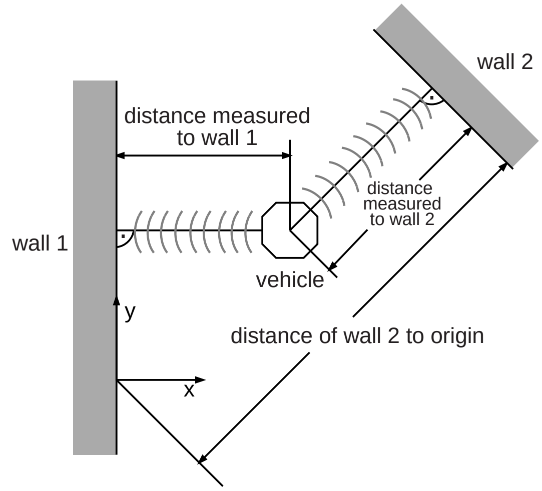

Figure 1: Setup for simulative example.

4 The New Filter

Now, we consider a combined uncertainty model [2, 3] with $v_k = e_k + c_k$. e_k is bounded according to $e_k^2 \leq E_k^y$ and c_k a zero mean, white, Gaussian random process with known variance according to $c_k \sim N(0,\ C_k^y)$.

The generalization of the measurement update is *not* simply a combination of the update formulae of Kalman filter and set theoretic filter. The update step conceptually is performed by intersecting two sets with random position. Of course, the update result is a complicated set with random size, orientation, and position. This exact result is approximated to second order, i.e., by an ellipsoidal set with a Gaussian distributed random midpoint. The mean of the midpoint is given by

$$\begin{aligned}\underline{x}_k = {} & \mathbf{W}_k^x \underline{x}_{k-1} + \underline{W}_k^y y_k \\ & - \left(\mathbf{W}_k^x \mathbf{C}_{k-1}\underline{H}_k - \underline{W}_k^y C_k^y\right) F_1\left(y_k - \underline{H}_k^T \underline{x}_{k-1}\right)\end{aligned} \tag{3}$$

with an associated covariance matrix

$$\begin{aligned}\mathbf{C}_k = {} & \mathbf{W}_k^x \mathbf{C}_{k-1}\left(\mathbf{W}_k^x\right)^T + \underline{W}_k^y\left(\underline{W}_k^y\right)^T C_k^y \\ & - \left(\mathbf{W}_k^x \mathbf{C}_{k-1}\underline{H}_k - \underline{W}_k^y C_k^y\right) \\ & \left(\mathbf{W}_k^x \mathbf{C}_{k-1}\underline{H}_k - \underline{W}_k^y C_k^y\right)^T F_2\left(y_k - \underline{H}_k^T \underline{x}_{k-1}\right) .\end{aligned} \tag{4}$$

$\mathbf{W}_k^x$, $\underline{W}_k^y$, $F_1\left(y_k - \underline{H}_k^T \underline{x}_{k-1}\right)$, $F_2\left(y_k - \underline{H}_k^T \underline{x}_{k-1}\right)$ are given in the appendix. For the set theoretic uncertainty we have

$$\begin{aligned}\mathbf{E}_k = {} & (1 + \lambda_k)\,\mathbf{E}_{k-1} \\ & - (1 + \lambda_k)\,\lambda_k \frac{\mathbf{E}_{k-1}\underline{H}_k\,\underline{H}_k^T\mathbf{E}_{k-1}}{E_k^y + \lambda_k\,\underline{H}_k^T \mathbf{E}_{k-1}\underline{H}_k} \ ,\end{aligned} \tag{5}$$

which defines the size and orientation of the ellipsoidal set. $\lambda_k \in [0, \infty)$ is chosen to minimize an appropriate function of $\mathbf{C}_k$ and $\mathbf{E}_k$.

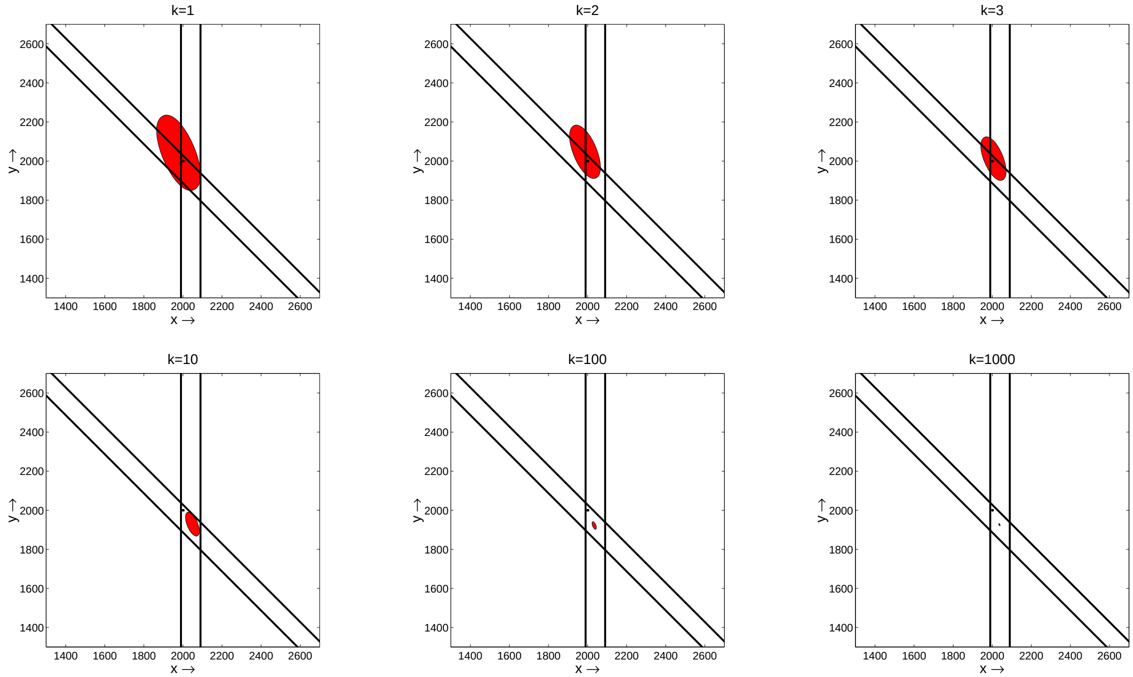

Figure 2: Results of Kalman filtering: Evolution of confidence sets over time.

Wall	1	2
Unit normal vector $\underline{H}_i$	$[1, 0]^T$	$-1/\sqrt{2}[1, 1]^T$
Nominal distance d_i	0	-6000
True distance $\tilde{d}_i$	-40	-6030
Bound b_i	50	50
Standard deviation σ_i	100	100

Table 1: Parameters of localization experiment.

5 Simulative Example

Consider a vehicle equipped with range sensors that measure the distances to two walls i, $i = 1, 2$, Figure 1. The wall positions are known within a given geometric tolerance, i.e.,

$$d_i = \tilde{d}_i + \Delta d_i \ , \text{ with } |\Delta d_i| \leq b_i \ ,$$

where $\tilde{d}_i$ denotes the unknown true (signed) distance of the wall to the origin and Δd_i is the unknown but bounded deviation of the nominal value d_i. The corresponding unit normal vector $\underline{H}_i$ is assumed to be known. The range measurements are corrupted by additive white Gaussian noise with zero mean and a variance σ_i^2. The measurement equation is given by

$$d_i + D_i^k = \underline{H}_i^T \tilde{\underline{x}} + \Delta d_i + c_i^k \ ,$$

where $c_i^k \sim N(0, \sigma_i)$, $\tilde{\underline{x}}$ denotes the vehicle position, and D_i is the measured distance. A true vehicle position $\tilde{\underline{x}} = [2000, 2000]^T$ is assumed. The remaining parameters are given in Tab. 1. The initial position is given by $\underline{x}_0 = [1900, 2100]^T$ with $\mathbf{E}_0 = \text{diag}\left(2000^2, 2000^2\right)$ and $\mathbf{C}_0 = \text{diag}\left(2000^2, \ 2000^2\right)$. At each time instant k, the distances to both walls are measured.

Results of the Kalman Filter: The Kalman filter is evaluated by recursively updating the position estimate using the equation for $\underline{x}_k$ in (1), $\mathbf{C}_k$ in (2). To employ standard Kalman filtering, the wall uncertainties are viewed as additional uncorrelated noise terms. For wall $i = 1, 2$, we obtain a total measurement variance of $C_k^y + E_k^y = \sigma_i^2 + b_i^2$. At every time step k, the filter is applied twice: The measured distance to wall 1 yields an intermediate estimate, and the measured distance to wall 2 yields the estimate $\underline{x}_k$ that incorporates all measurements available up to time k. The evolution of the resulting confidence set is depicted in Figure 2 for $k = 1, 2, 3, 10, 100, 1000$. The optimal estimate for an infinite number of measurements would be the set resulting from intersecting the two strips that correspond to the uncertainty of the two walls. The exact state $\tilde{\underline{x}} = [2000, 2000]^T$ is marked by a dot. Here, the confidence set has been calculated based on 9 times the Kalman filter covariance matrix $\mathbf{C}_k$ centered at $\underline{x}_k$. **Note:** The confidence set for $k \to \infty$ does *not* contain the true state.

Results of the Set Theoretic Estimator: The set theoretic estimator is applied in a similar fashion. Here, the noise in the distance measurements is included by increasing the measurement uncertainty according to $E_k^y + \gamma\, C_k^y = b_i^2 + \gamma\, \sigma_i^2$, $i = 1, 2$. To consider the long tails of the assumed Gaussian distribution,

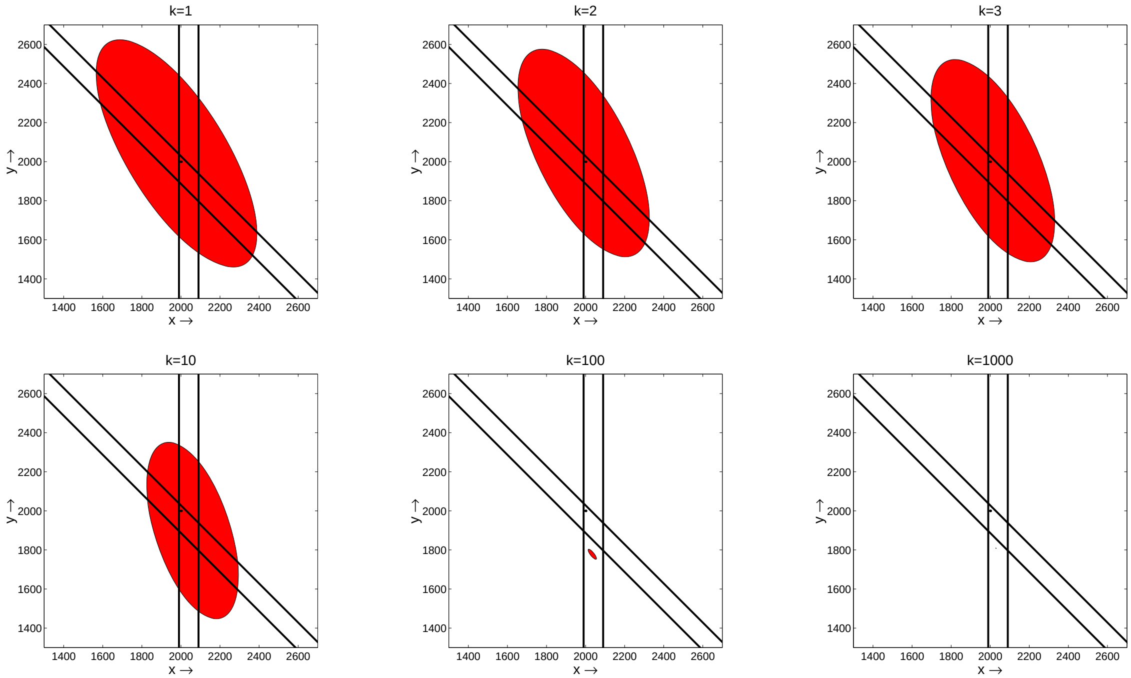

Figure 3: Results of set theoretic filtering ($E_k^y + 4\,C_k^y$): Evolution of confidence sets over time.

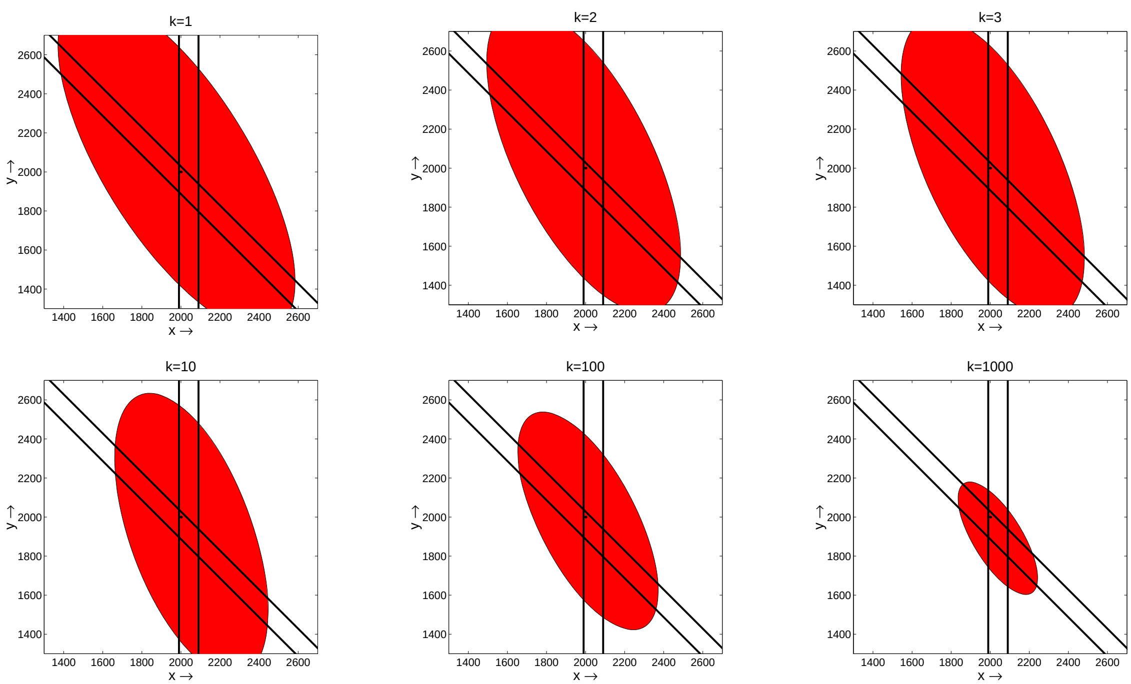

Figure 4: Results of set theoretic filtering ($E_k^y + 9\,C_k^y$): Evolution of confidence sets over time.

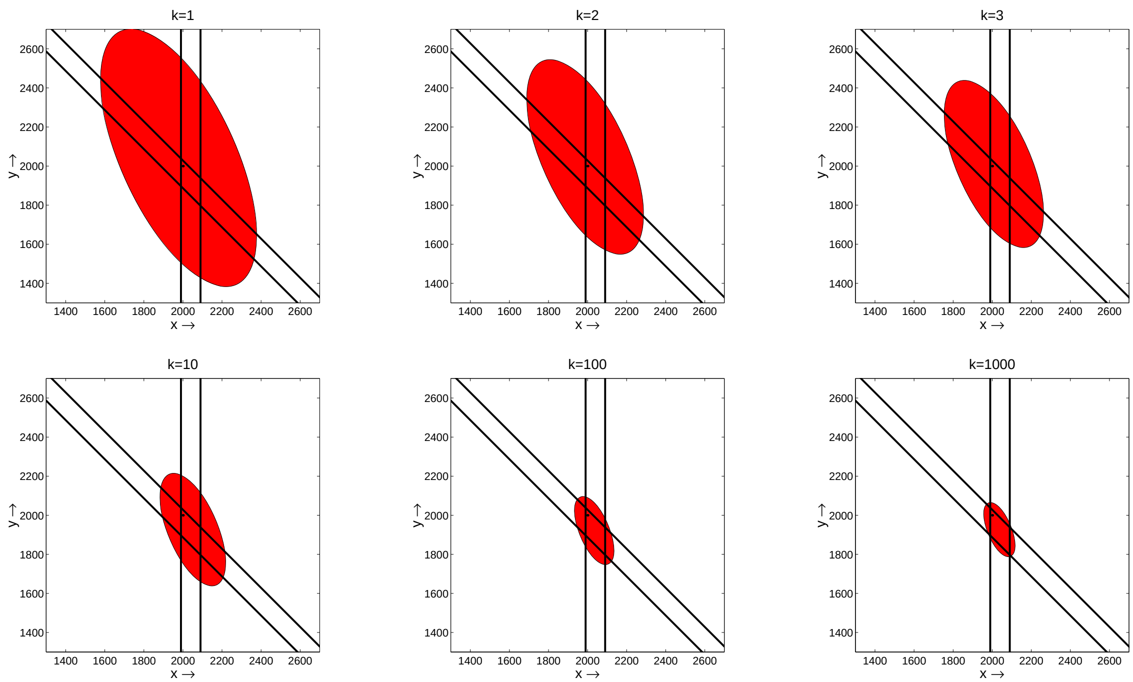

Figure 5: Results of applying the new estimator: Evolution of confidence sets over time.

γ must be chosen very conservative. The smaller the choice of γ, the higher the probability of producing a wrong estimate.

The result for $\gamma = 4$ is shown in Figure 3. **Note:** The confidence set for $k \to \infty$ does *not* contain the true state. Application of $\gamma = 9$ is shown in Figure 4.

Results of the New Estimator: The proposed new estimator is evaluated by recursively updating the position estimate using the equation for $\underline{x}_k$ in (3), $\mathbf{E}_k$ in (5), and $\mathbf{C}_k$ in (4) twice: Once for wall 1 with $E_k^y = b_1^2$ and $C_k^y = \sigma_1^2$, which yields an intermediate estimate, and once for wall 2 with $E_k^y = b_2^2$ and $C_k^y = \sigma_2^2$, which yields the estimate $\underline{x}_k$ that incorporates all measurements available up to time k. The parameter λ_k is chosen such that $|\mathbf{E}_k| + |\mathbf{C}_k|$ is minimized. Figure 5 depicts how the resulting estimate evolves over time. The confidence set is given as the Minkowski sum of $\mathbf{E}_k$ and $9\,\mathbf{C}_k$ centered at $\underline{x}_k$. **Note:** The confidence set for $k \to \infty$ bounds the exact set from above and hence contains the true state.

6 Conclusions

A vast class of estimation problems can be attacked as a mixed noise problem, i.e., the arising uncertainties can be modeled as being additively composed of both 1) noise with known distribution and 2) noise with known bounds. For these problems, a new estimator has been derived for the important case of linear systems with arbitrary dimensional states and scalar measurements. The estimator provides solution sets with Gaussian distributed random positions. Of course, the new estimator contains the Kalman filter and the set theoretic filter as border cases.

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7 Appendix

For the weighting factors $\mathbf{W}_k^x$, $\underline{W}_k^y$ we have

$$\mathbf{W}_k^x = I - \lambda_k \frac{\mathbf{E}_{k-1} \underline{H}_k \underline{H}_k^T}{E_k^y + \lambda_k \underline{H}_k^T \mathbf{E}_{k-1} \underline{H}_k} \quad ,$$
$$\underline{W}_k^y = \frac{\lambda_k \mathbf{E}_{k-1} \underline{H}_k}{E_k^y + \lambda_k \underline{H}_k^T \mathbf{E}_{k-1} \underline{H}_k} \quad .$$

The nonlinear functions $F_1\left(y_k - \underline{H}_k^T \underline{x}_{k-1}\right)$, $F_2\left(y_k - \underline{H}_k^T \underline{x}_{k-1}\right)$ of the innovation $y_k - \underline{H}_k^T \underline{x}_{k-1}$ are given by

$$F_1\left(y_k - \underline{H}_k^T \underline{x}_{k-1}\right) = G_0\left(y_k - \underline{H}_k^T \underline{x}_{k-1}, \sqrt{E_k^y} + \sqrt{\underline{H}_k^T \mathbf{E}_{k-1} \underline{H}_k}, \sqrt{\underline{H}_k^T \mathbf{C}_{k-1} \underline{H}_k + C_k^y}\right) \quad ,$$
$$F_2\left(y_k - \underline{H}_k^T \underline{x}_{k-1}\right) = \left[G_0\left(y_k - \underline{H}_k^T \underline{x}_{k-1}, \sqrt{E_k^y} + \sqrt{\underline{H}_k^T \mathbf{E}_{k-1} \underline{H}_k}, \sqrt{\underline{H}_k^T \mathbf{C}_{k-1} \underline{H}_k + C_k^y}\right)\right]^2 + \frac{G_1\left(y_k - \underline{H}_k^T \underline{x}_{k-1}, \sqrt{E_k^y} + \sqrt{\underline{H}_k^T \mathbf{E}_{k-1} \underline{H}_k}, \sqrt{\underline{H}_k^T \mathbf{C}_{k-1} \underline{H}_k + C_k^y}\right)}{\underline{H}_k^T \mathbf{C}_{k-1} \underline{H}_k + C_k^y}$$

with functions G_0 and G_1, where the erf-function is defined according to [14],

$$G_0(x, B, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \frac{\exp\left\{-\frac{1}{2}\frac{(x-B)^2}{\sigma^2}\right\} - \exp\left\{-\frac{1}{2}\frac{(x+B)^2}{\sigma^2}\right\}}{\operatorname{erf}\left\{\frac{x-B}{\sigma}\right\} - \operatorname{erf}\left\{\frac{x+B}{\sigma}\right\}} \quad ,$$
$$G_1(x, B, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \frac{(x-B)\exp\left\{-\frac{1}{2}\frac{(x-B)^2}{\sigma^2}\right\} - (x+B)\exp\left\{-\frac{1}{2}\frac{(x+B)^2}{\sigma^2}\right\}}{\operatorname{erf}\left\{\frac{x-B}{\sigma}\right\} - \operatorname{erf}\left\{\frac{x+B}{\sigma}\right\}} \quad .$$