Gaussianity-Preserving Event-Based State Estimation with an FIR-Based Stochastic Trigger

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***Abstract*—With modern communication technology, sensors, estimators, and controllers can be pushed apart to distribute intelligence over wide distances. Instead of congesting channels by periodic data transmissions, smart sensors can decide on their own whether data are worth transmitting. This paper studies event-based transmissions from sensor to estimator. The sensor-side event trigger conveys usable information even if no transmission is triggered. In the absence of data, such implicit information can still be exploited by the remote Kalman filter. For this purpose, an easy-to-implement triggering mechanism is proposed based on a Finite Impulse Response prediction that is compared against a stochastic decision variable. By the aid of the stochastic event trigger, the implicit information retains a Gaussian representation and can easily be processed by the Kalman filter. The parameters for the stochastic trigger are retrieved from the Finite Impulse Response filter, which contributes to reducing the communication rate significantly, as shown in simulations.**

***Index Terms*—Kalman filtering, estimation, sensor networks**

I. INTRODUCTION

WITH cheap integrated sensors and wireless communication infrastructures, data can be acquired ubiquitously and pervasively. Effective resource allocation in wireless networks has to address the question of when and how often data must be transmitted to remote processing units. Recent solutions to this question indicate a paradigm shift from time-periodic to data-driven or event-based transmission schedules. Special focus is currently placed on state estimation with event-triggered sensor data [1]–[3]. Sophisticated event-based schedules assess the utility of sensor data to trigger a transmission to the remote estimator. Prominent examples are the predicted error variance of the measurements [4] or send-on-delta schemes [5]. Finding a trade-off between estimation quality and communication rate [6] poses a significant challenge in applying event-based estimation. For this reason, a joint objective function that comprises both the estimation error and a communication penalty is considered in [7]. Accordingly, transmitting sensor readings less frequently may endanger the stability of the remote estimator [8], [9]. However, overcoming these hindrances rewards us with the opportunity to implement data-efficient algorithms for estimation and control already at the sensor—to operate low-power sensor modules in future low-power wide area networks effectively.

The design of event triggering criteria is central to reducing communication rate and preserving estimation accuracy at the same time. For instance, a study of different event-triggering criteria with respect to estimation error and communication resources has been conducted in [10]. To retrieve the most information out of an event-based sampling of the sensor output, the remote estimator can exploit the fact that no transmission was triggered. In doing so, the absence of sensor data can still be fed into the remote estimator as implicit—also called negative—measurement information [11], [12]. The estimator can hence perform time-periodic measurement updates [2] though data are only available aperiodically. Typical threshold-based, deterministic triggers relate the implicit measurements to bounds on the actual sensor signals. For these triggers, hybrid estimation concepts are used [11]–[13] that combine Kalman filtering (KF) with ellipsoidal calculus [14]. In [15] and [16], similar bounds for deterministic triggers are incorporated in distributed Kalman filter schemes to fuse estimates. However, the set-theoretic property of deterministic event triggers entails the disadvantage of a more complicated estimator design due to the involved nonlinearity.

As an alternative to a deterministic design, stochastic triggers have been proposed that preserve the Gaussianity of the implicit measurement information. Hence, such measurements can be used directly in a standard Kalman filter. In [17], open- and closed-loop trigger mechanisms have been studied while the latter require a feedback from the remote estimator to the sensor. These concepts have stimulated further research, e.g., in the context of multi-sensor data fusion [18] or with respect to the information form of the Kalman filter [19]. By introducing a local prediction [20] at the sensor, the communication rate can be further reduced and the information gain can be increased. In [21], the sensor runs a local Kalman filter to generate a triggering decision, which is based on comparing the current estimate to the prediction from the last event. In case of a triggered transmission, the remote estimator replaces its own result by the received estimate of the sensor.

In this paper, the sensor follows a transmission schedule that is independently designed from the remote estimator. An unbiased Finite Impulse Response (FIR) filter [22] is employed by the sensor to predict measurement outcomes that are compared against the actual sensor readings in order to determine an event trigger. The sensor shares two pieces of information with the remote Kalman filter when an event is triggered: the current sensor value as well as the FIR estimate. The latter information is exploited by the remote estimator to

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compute implicit measurements for the next time steps when no event will be triggered. Due to a stochastic decision variable, the implicit measurements remain Gaussian and can easily be incorporated by the remote Kalman filter. Compared with other trigger concepts, this approach offers several advantages: 1) The sensor does not need access to prior information about the state. 2) No feedback from the estimator to the sensor is required. 3) The trigger inherits the robustness of the FIR filter against unmodeled disturbances. This robustness leads to both lower transmission rates and higher accuracy of the remote estimator. 4) The proposed event-based estimator can be applied to unstable systems and reaches the performance of the closed-loop schedule even for short horizons of the FIR filter. In the following, the proposed event-based estimator is derived, and these properties are discussed in four simulations.

II. PROBLEM FORMULATION

A. System Model

A discrete-time linear system is considered given by

$$\underline{\mathbf{x}}_{k+1} = \mathbf{A}\,\underline{\mathbf{x}}_k + \underline{\mathbf{w}}_k\,, \quad \underline{\mathbf{y}}_k = \mathbf{C}\,\underline{\mathbf{x}}_k + \underline{\mathbf{v}}_k\,,$$

where $\underline{\mathbf{x}}_k \in \mathbb{R}^{n_x}$ is the state at time step $k \in \mathbb{N}$, and $\underline{\mathbf{y}}_k \in \mathbb{R}^{n_y}$ denotes the observation. The time-invariant process and measurement matrices are given by $\mathbf{A} \in \mathbb{R}^{n_x \times n_x}$ and $\mathbf{C} \in \mathbb{R}^{n_y \times n_x}$, respectively, and the pair $(\mathbf{A}, \mathbf{C})$ is detectable. The process noise $\underline{\mathbf{w}}_m \sim \mathcal{N}(\underline{0}, \mathbf{W})$ and measurement noise $\underline{\mathbf{v}}_n \sim \mathcal{N}(\underline{0}, \mathbf{R})$ are white and mutually uncorrelated for arbitrary $m, n \in \mathbb{N}$. $\mathbf{A}$, $\mathbf{R}$, and $\mathbf{W}$ are non-singular.

In order to preserve the Gaussianity of the state variable despite the use of event-based triggering, a stochastic trigger as presented by [17] is employed. The principles of stochastic triggering are explained in the following section.

B. Stochastic Triggering

Let γ_k be the decision variable at time step k, then $\gamma_k = 1$ denotes that an event is triggered, and $\underline{\mathbf{y}}_k$ is sent to the receiver. For $\gamma_k = 0$, no transmission is triggered. To determine γ_k, an independently and identically distributed random variable ξ is generated, which is uniformly distributed over $[0, 1]$. The decision scheme is given by

$$\gamma_k = \begin{cases} 1, & \xi_k > \phi(\underline{\mathbf{z}}_k)\,, \\ 0, & \xi_k \leq \phi(\underline{\mathbf{z}}_k)\,, \end{cases} \tag{1}$$

where $\phi(\underline{\mathbf{z}}_k) = \exp\left(-\frac{1}{2}\,\underline{\mathbf{z}}_k^{\mathrm{T}}\,\mathbf{Z}^{-1}\,\underline{\mathbf{z}}_k\right)$. The matrix $\mathbf{Z}$ is a design parameter and takes the role of an additive measurement uncertainty as shown subsequently. Due to the design of $\phi(\underline{\mathbf{z}}_k)$ and the properties of ξ_k, the sending probability given $\underline{\mathbf{z}}_k$ yields

$$\Pr\left\{\gamma_k = 1 \,\middle|\, \underline{\mathbf{z}}_k\right\} = 1 - \phi(\underline{\mathbf{z}}_k)\,,$$
$$\Pr\left\{\gamma_k = 0 \,\middle|\, \underline{\mathbf{z}}_k\right\} = \phi(\underline{\mathbf{z}}_k)\,.$$

Thus, the choice of $\underline{\mathbf{z}}_k$ is crucial and is the key design feature of the stochastic trigger. Usually, $\underline{\mathbf{z}}_k$ depends on the current measurement $\underline{\mathbf{y}}_k$ at time step k. Multiple choices of $\underline{\mathbf{z}}_k$ and respective transmission schemes have been presented in literature, each one pursuing a different objective. Two strategies were proposed by Han et al. [17] and are briefly explained in the following.

- The *Open Loop* (OL) scheme where $\underline{\mathbf{z}}_k = \underline{\mathbf{y}}_k$: This is the simplest possible scheme only depending on the current measurement $\underline{\mathbf{y}}_k$. Drawbacks are the low efficiency, i.e., the communication rate cannot be reduced significantly for given error bounds, and the limitation to stable systems with zero mean.
- The *Closed Loop* (CL) scheme where $\underline{\mathbf{z}}_k = \underline{\mathbf{y}}_k - \hat{\underline{\mathbf{y}}}_k^-$: At every time instance the sensor receives the predicted measurement $\hat{\underline{\mathbf{y}}}_k^-$ as a feedback from the receiver. This leads to a very efficient trigger as the sensor has information about the estimation quality at the receiver but requires an undesirable bidirectional communication.

Two more schemes were proposed by Andren and Cervin [20]:

- The *Stochastic Send-On-Delta* (SSOD) scheme where $\underline{\mathbf{z}}_k = \underline{\mathbf{y}}_k - \underline{\mathbf{y}}_{k-l}$: The last measurement $\underline{\mathbf{y}}_{k-l}$ sent l time steps ago is subtracted from the current measurement. The scheme resembles the deterministic *Send-On-Delta* explained, e.g., in [5], [13] and leads to a high triggering probability if the measurements strongly vary. However, no system behavior is incorporated and therefore the scheme is inefficient for highly dynamic systems.
- The *Stochastic Send-On-Delta with Simple Prediction* (SSODP) scheme where $\underline{\mathbf{z}}_k = \underline{\mathbf{y}}_k - \mathbf{S}_l \underline{\mathbf{y}}_{k-l}$: This extension of SSOD uses the steady-state system behavior to calculate weights $\mathbf{S}_l$ to predict $\underline{\mathbf{y}}_{k-l}$ up to the current time step. As an unstable system does not reach a steady state, this method is only applicable for stable systems.

To overcome the limitations of the aforementioned methods, a novel stochastic trigger is developed in this work.

III. FIR-BASED STOCHASTIC TRIGGER

A. Triggering Mechanism

As identified before, the SSODP scheme shall be extended to cover unstable systems. The SSODP scheme intends to determine the steady-state covariance $\mathbf{\Sigma}$ of the state vector and uses the discrete-time Lyapunov equation to derive the weights $\mathbf{S}_l$ mentioned above:

$$\mathbf{\Sigma} = \mathbf{A}\,\mathbf{\Sigma}\,\mathbf{A}^{\mathrm{T}} + \mathbf{W}\,,$$
$$\mathbf{S}_l = \mathbf{C}\,\mathbf{A}^l\,\mathbf{\Sigma}\,\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\,\mathbf{\Sigma}\,\mathbf{C}^{\mathrm{T}} + \mathbf{R}\right)^{-1}\,.$$

Since no steady state exists for unstable systems, $\mathbf{\Sigma}$ cannot be calculated in this case. Alternatively, the time-variant estimation error covariance $\mathbf{P}_k$ of the remote estimate could be used. Unfortunately, $\mathbf{P}_k$ is not known at the sensor unless bidirectional communication between sensor and receiver is used or a second, simultaneous KF (sKF) at the sensor node is implemented. Both cases are not favorable; therefore, a more sophisticated solution is presented in the following.

To be efficient in terms of communication rate versus estimation error, a triggering unit located at the sensor node must be able to anticipate the estimation quality at the receiver based on the present information. Because a unidirectional communication is favored and the computational power at the sensor is supposed to be limited, a simple solution with sensor-only information is used in this work. The proposed *Finite Impulse Response Prediction* scheme is based on the unbiased FIR filter presented by Shmaliy et al. [22]. This filter does not need knowledge about the prior state, the error statistics, or the entire history. However, as error statistics are known in our scenario, they will be used for enhanced stability of the proposed method. The foundations of the novel FIR-based stochastic trigger (SSODP-FIR) are explained in the following, and its advantages over Kalman filter-based triggers are discussed in Sec. IV.

First of all, the last m measurements are collected in a column vector

$$\underline{\mathbf{y}}_{\text{last}} = \begin{pmatrix} \underline{\mathbf{y}}_k \\ \underline{\mathbf{y}}_{k-1} \\ \vdots \\ \underline{\mathbf{y}}_{k-(m-1)} \end{pmatrix} = \begin{pmatrix} \mathbf{C} \\ \mathbf{C}\,\mathbf{A}^{-1} \\ \vdots \\ \mathbf{C}\,\mathbf{A}^{-(m-1)} \end{pmatrix} \cdot \underline{\mathbf{x}}_k + \underline{\mathbf{n}}_k = \mathbf{M} \cdot \underline{\mathbf{x}}_k + \underline{\mathbf{n}}_k \,,$$

and expressed in terms of the current state $\underline{\mathbf{x}}_k$. The number m is chosen such that $\mathbf{M}$ has full column rank. More precisely, the measurement $\underline{\mathbf{y}}_{k-l}$ for $l \in \{1, \ldots, m-1\}$ is related to the retrodicted state through

$$\begin{aligned} \underline{\mathbf{y}}_{k-l} &= \mathbf{C}\underline{\mathbf{x}}_{k-l} + \underline{\mathbf{v}}_{k-l} \\ &= \mathbf{C}\mathbf{A}^{-1}\big(\underline{\mathbf{x}}_{k-l+1} - \underline{\mathbf{w}}_{k-l+1}\big) + \underline{\mathbf{v}}_{k-l} = \ldots \\ &= \mathbf{C}\mathbf{A}^{-l}\underline{\mathbf{x}}_k - \sum_{i=1}^{l-1} \mathbf{C}\mathbf{A}^{-(l-i)}\underline{\mathbf{w}}_{k-i} + \underline{\mathbf{v}}_{k-l} \,. \end{aligned}$$

The additive error $\underline{\mathbf{n}}_k$ covers the process and measurement noise terms of the retrodicted state and is hence given by

$$\underline{\mathbf{n}}_k = -\begin{pmatrix} \underline{\mathbf{0}} \\ \mathbf{C}\mathbf{A}^{-1}\underline{\mathbf{w}}_{k-1} \\ \vdots \\ \sum\limits_{i=1}^{m-1} \mathbf{C}\mathbf{A}^{-(m-i)}\underline{\mathbf{w}}_{k-i} \end{pmatrix} + \begin{pmatrix} \underline{\mathbf{v}}_k \\ \underline{\mathbf{v}}_{k-1} \\ \vdots \\ \underline{\mathbf{v}}_{k-(m-1)} \end{pmatrix} .$$

It has zero mean and the non-singular covariance matrix

$$\mathbf{V} = \mathrm{Cov}\{\underline{\mathbf{n}}_k\} \tag{2}$$

with the block entries[1]

$$(\mathbf{V})_{pq} = \mathbf{C}\left(\sum_{i=1}^{\min(p,q)-1} \mathbf{A}^{-(p-i)}\mathbf{W}\big(\mathbf{A}^{-(q-i)}\big)^{\mathrm{T}}\right)\mathbf{C}^{\mathrm{T}} + \delta_{pq}\mathbf{R} \,,$$

where δ_{pq} is the Kronecker delta and $p, q \in \{1, \ldots, m\}$. The joint covariance matrix $\mathbf{V}$ characterizes the correlations between subsequent measurements due to process and measurement noise. In doing so, we arrive at an uncertainty-aware FIR filter, which solves the corresponding weighted least squares problem $\arg\min_{\underline{x}} \|\underline{\mathbf{y}}_{\text{last}} - \mathbf{M}\,\underline{x}_k\|^2_{\mathbf{V}}$ by

$$\tilde{\underline{\mathbf{x}}}_k^{\mathrm{FIR}} = (\mathbf{M}^{\mathrm{T}}\,\mathbf{V}^{-1}\,\mathbf{M})^{-1}\,\mathbf{M}^{\mathrm{T}}\,\mathbf{V}^{-1}\,\underline{\mathbf{y}}_{\text{last}} \,. \tag{3}$$

Like the Kalman filter estimate, the least squares estimate $\tilde{\underline{\mathbf{x}}}_k^{\mathrm{FIR}}$ is unbiased [22]. Thus, the estimation error is given by

$$\underline{\boldsymbol{\epsilon}} = (\mathbf{M}^{\mathrm{T}}\,\mathbf{V}^{-1}\,\mathbf{M})^{-1}\,\mathbf{M}^{\mathrm{T}}\,\mathbf{V}^{-1}\underline{\mathbf{n}}_k$$

and the corresponding estimation error covariance by

$$\mathbf{E} = \mathrm{Cov}\{\underline{\boldsymbol{\epsilon}}\} = (\mathbf{M}^{\mathrm{T}}\,\mathbf{V}^{-1}\,\mathbf{M})^{-1} \,. \tag{4}$$

Both the measurement $\underline{\mathbf{y}}_k$ and the estimate $\tilde{\underline{\mathbf{x}}}_k^{\mathrm{FIR}}$ are sent to the receiver if a transmission event is triggered.

In the next time step $k+1$, the estimate $\tilde{\underline{\mathbf{x}}}_k^{\mathrm{FIR}}$ is predicted according to the system model and $\underline{\mathbf{z}}_{k+1}$ is calculated by

$$\hat{\underline{\mathbf{y}}}_{k+1}^{\mathrm{FIR}} = \mathbf{C}\,\mathbf{A}\,\tilde{\underline{\mathbf{x}}}_k^{\mathrm{FIR}} \,, \tag{5}$$

$$\underline{\mathbf{z}}_{k+1} = \underline{\mathbf{y}}_{k+1} - \hat{\underline{\mathbf{y}}}_{k+1}^{\mathrm{FIR}} \,. \tag{6}$$

Consequently, $\underline{\mathbf{z}}_{k+1}$ compares the actual and the expected measurement predicted from the FIR estimate. With $\phi(\underline{\mathbf{z}}_{k+1})$, a trigger decision is reached according to eq. (1). The following two cases need to be considered:

$\gamma_{k+1} = 1$: In case of a transmission, the sender computes a new FIR estimate $\tilde{\underline{\mathbf{x}}}_{k+1}^{\mathrm{FIR}}$ by means of eq. (3) using its buffered measurements. The current measurement and this FIR estimate are sent to the receiver.

$\gamma_{k+1} = 0$: No transmission is triggered. The FIR estimates at the sender and receiver are predicted and remain the same on both sides, i.e., $\tilde{\underline{\mathbf{x}}}_{k+1}^{\mathrm{FIR}} = \mathbf{A}\,\tilde{\underline{\mathbf{x}}}_k^{\mathrm{FIR}}$. The receiver uses its FIR estimate to compute (5), which serves as implicit measurement information.

These computations are repeated for $k+2, k+3, \ldots$. In the following subsection, it is shown how the receiver can exploit the explicit and implicit information in a KF.

B. State Estimation

To incorporate implicit measurement information, the Kalman filter equations have to be modified correspondingly. Deterministic triggers have been studied, e.g., in [1], which require to define bounds on the error covariances. Following the considerations in [20], the modified Kalman filter equations for stochastic triggers are similar but provide the exact covariances. In order to use the stochastic trigger (1), we consider $\underline{\mathbf{z}}_k = \underline{\mathbf{y}}_k - \underline{\mathbf{c}}_k$ with $\underline{\mathbf{c}}_k$ being an arbitrary vector. The remote Kalman filter computes estimate $\hat{\underline{\mathbf{x}}}_k$ and covariance $\mathbf{P}_k$ according to

$$\hat{\underline{\mathbf{x}}}_k^{-} = \mathbf{A}\,\hat{\underline{\mathbf{x}}}_{k-1} \,, \tag{7}$$

$$\mathbf{P}_k^{-} = \mathbf{A}\,\mathbf{P}_{k-1}\,\mathbf{A}^{\mathrm{T}} + \mathbf{W} \,. \tag{8}$$

in the prediction step and

$$\hat{\underline{\mathbf{x}}}_k = \hat{\underline{\mathbf{x}}}_k^{-} + \mathbf{K}_k\left(\gamma_k\,\underline{\mathbf{z}}_k - \hat{\underline{\mathbf{z}}}_k^{-}\right) , \tag{9}$$

$$\underline{\mathbf{z}}_k = \underline{\mathbf{y}}_k - \underline{\mathbf{c}}_k \,, \quad \hat{\underline{\mathbf{z}}}_k^{-} = \mathbf{C}\,\hat{\underline{\mathbf{x}}}_k^{-} - \underline{\mathbf{c}}_k \,, \tag{10}$$

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\,\mathbf{C})\,\mathbf{P}_k^{-} \,, \tag{11}$$

$$\mathbf{K}_k = \mathbf{P}_k^{-}\,\mathbf{C}^{\mathrm{T}}\,\left[\mathbf{C}\,\mathbf{P}_k^{-}\,\mathbf{C}^{\mathrm{T}} + \mathbf{R} + (1-\gamma_k)\,\mathbf{Z}\right]^{-1} \,, \tag{12}$$

in the filtering step, which depends on the decision γ_k.

In case of $\gamma_k = 0$, the innovation becomes $(\underline{\mathbf{c}}_k - \mathbf{C}\,\hat{\underline{\mathbf{x}}}_k^{-})$, which means that $\underline{\mathbf{c}}_k$ serves as an *implicit* measurement. For the proposed FIR-based trigger mechanism (6), the implicit measurement is hence given by (5), i.e., $\underline{\mathbf{c}}_k = \mathbf{C}\,\mathbf{A}\,\tilde{\underline{\mathbf{x}}}_{k-1}^{\mathrm{FIR}}$. The gist of this concept is that the remote KF can compute the implicit measurement on its own: For any sequence $\gamma_{k+l} = \ldots = \gamma_{k+1} = \gamma_k = 0$ of absent events, $\underline{\mathbf{c}}_{k+l} = \mathbf{C}\,\mathbf{A}^{l+1}\,\tilde{\underline{\mathbf{x}}}_{k-1}^{\mathrm{FIR}}$

[1] For $\sum_{i=1}^{0}$, the sum vanishes.

is employed. A proof to justify this construction with an arbitrary choice of $\underline{\mathbf{c}}_k$ is given in the Appendix.

Remark. The proposed concept requires that both the measurement and the FIR estimate are sent to the receiver when events are triggered. These values share redundant information as the FIR estimate already incorporates the current measurement. However, if one would use the FIR state estimate instead of the real measurement in the Kalman filter also for $\gamma_k = 1$, the conditional independence in the measurement update is violated. To obtain a Minimum Mean Square Error (MMSE) estimator using only the FIR state estimate, a Colored Noise KF has to be designed. This task is left for future work.

C. Communication Rate

In the proposed triggering scheme, different estimators are employed at the sensor and the receiver. The sensor uses an FIR estimator, the receiver a modified Kalman filter. This needs to be taken into account for the calculation of the asymptotic expected estimation error covariance bounds, $\lim_{k\to\infty} \mathrm{E}\{\mathbf{P}_k^-\}$, and the communication rate.

1) Covariance Bounds: As explained in [17], the lower bound $\underline{\mathbf{X}}^{\mathrm{KF}}$ and the upper bound $\overline{\mathbf{X}}^{\mathrm{KF}}$ of the expected estimation error covariance of the Kalman filter at the receiver are given by the solutions of the Algebraic Riccati equations

$$\underline{\mathbf{X}}^{\mathrm{KF}} = \mathbf{A}\underline{\mathbf{X}}^{\mathrm{KF}}\mathbf{A}^{\mathrm{T}} + \mathbf{W} - \mathbf{A}\underline{\mathbf{X}}^{\mathrm{KF}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\underline{\mathbf{X}}^{\mathrm{KF}}\mathbf{C}^{\mathrm{T}} + \mathbf{R}\right)^{-1}\mathbf{C}\underline{\mathbf{X}}^{\mathrm{KF}}\mathbf{A}^{\mathrm{T}}, \tag{13}$$

$$\overline{\mathbf{X}}^{\mathrm{KF}} = \mathbf{A}\overline{\mathbf{X}}^{\mathrm{KF}}\mathbf{A}^{\mathrm{T}} + \mathbf{W} - \mathbf{A}\overline{\mathbf{X}}^{\mathrm{KF}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\overline{\mathbf{X}}^{\mathrm{KF}}\mathbf{C}^{\mathrm{T}} + \mathbf{R} + \mathbf{Z}\right)^{-1}\mathbf{C}\overline{\mathbf{X}}^{\mathrm{KF}}\mathbf{A}^{\mathrm{T}}, \tag{14}$$

respectively. This holds, because the form of the Kalman filter equations (7)–(12) does not depend on the specific choice of $\underline{\mathbf{c}}_k$, as discussed in the Appendix. The lower and upper bounds correspond to the extreme cases $\gamma_k = 1, \forall k$ and $\gamma_k = 0, \forall k$, respectively. In the latter case, the additional implicit measurement noise $\mathbf{Z}$ comes into play.

2) Communication Rate: By utilizing the results in [17], the communication rate can be determined according to

$$\gamma = 1 - \frac{1}{\sqrt{\det(\mathbf{I} + \mathbf{\Pi}\,\mathbf{Z}^{-1})}},$$

where $\mathbf{\Pi} = \lim_{k\to\infty} \mathrm{E}\{\mathrm{Cov}\{\underline{\mathbf{z}}_k\}\}$ denotes the asymptotic covariance of $\underline{\mathbf{z}}_k$ with the expectation over all possible trigger decisions. Unless the OL-scheme is used, determining $\mathbf{\Pi}$ is not trivial. Therefore, a similar approach as in [17] is chosen and upper and lower bounds of the communication rate are specified in the following.

Since the SSODP-FIR scheme does not include any receiver-to-sensor communication, the communication rate only depends on the FIR estimation performed at the sensor. By reusing equations (2) and (4) and accounting for the implicit measurements used in non-sending instances, we obtain the following lower and upper bounds for $\mathbf{X}^{\mathrm{FIR}}$:

$$\underline{\mathbf{X}}^{\mathrm{FIR}} = \mathbf{A}\,(\mathbf{M}^{\mathrm{T}}\,\mathbf{V}^{-1}\,\mathbf{M})^{-1}\,\mathbf{A}^{\mathrm{T}} + \mathbf{W}, \tag{15}$$

$$\overline{\mathbf{X}}^{\mathrm{FIR}} = \mathbf{A}\,(\mathbf{M}^{\mathrm{T}}\,\overline{\mathbf{V}}^{-1}\,\mathbf{M})^{-1}\,\mathbf{A}^{\mathrm{T}} + \mathbf{W}, \tag{16}$$

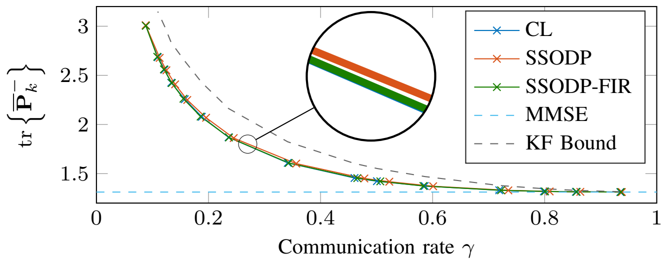


Fig. 1: Error covariance over communication rate for the scalar system using CL, SSODP and SSODP-FIR.

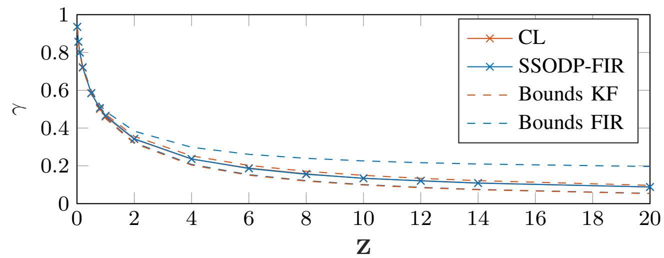


Fig. 2: Communication rate over $\mathbf{Z}$ for the scalar system using CL and SSODP-FIR.

with $\overline{\mathbf{V}} = \mathbf{V} + \mathrm{diag}\{\mathbf{Z}, \ldots, \mathbf{Z}\}$, which correspond to $\gamma_k = 1, \forall k$ and $\gamma_k = 0, \forall k$, respectively. From the estimation error covariance $\mathbf{X}^{\mathrm{FIR}}$, the covariance of $\underline{\mathbf{z}}_k$ can be determined by

$$\mathbf{\Pi} = \mathbf{C}\,\mathbf{X}^{\mathrm{FIR}}\,\mathbf{C}^{\mathrm{T}} + \mathbf{R}.$$

Hence, the lower and upper bounds of the communication rate are given by

$$\underline{\gamma} = 1 - \frac{1}{\sqrt{\det\left(\mathbf{I} + \left(\mathbf{C}\,\underline{\mathbf{X}}^{\mathrm{FIR}}\,\mathbf{C}^{\mathrm{T}} + \mathbf{R}\right)\mathbf{Z}^{-1}\right)}},$$

$$\overline{\gamma} = 1 - \frac{1}{\sqrt{\det\left(\mathbf{I} + \left(\mathbf{C}\,\overline{\mathbf{X}}^{\mathrm{FIR}}\,\mathbf{C}^{\mathrm{T}} + \mathbf{R}\right)\mathbf{Z}^{-1}\right)}}.$$

In the following section, the beneficial aspects of the SSODP-FIR scheme will be pointed out with the help of four simulation examples.

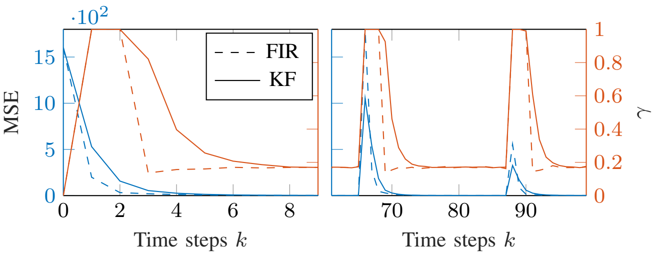
IV. SIMULATION EXAMPLES

A. Scalar Stable System

Two stable systems as found in [17], [20] are simulated to show the potential of the novel triggering scheme. As a simple example a first-order stable system is analyzed:

$$\mathbf{A} = 0.95, \quad \mathbf{W} = 0.8, \quad \mathbf{C} = 1, \quad \mathbf{R} = 1.$$

In Fig. 1, the expected error covariance $\overline{\mathbf{P}}^- = \lim_{k\to\infty} \mathrm{E}\{\mathbf{P}_k^-\}$ is plotted over the communication rate and is compared for different triggering schemes, namely CL, SSODP, and SSODP-FIR. The number of previous measurements to calculate the FIR estimate is set to $m = 2$. To achieve different communication rates, $\mathbf{Z}$ is varied between 0.01 and 20 in each example. The presented results have been obtained by averaging over $50\,000$ runs. As expected, all three schemes perform very well in this scenario; a close look reveals that the curves of CL and SSODP-FIR are almost indistinguishable and slightly lower than SSODP for all communication rates. The dashed lines



(a) Initialization offset.

(b) Noise peaks.

Fig. 3: Impact of unmodeled noise for the scalar system using SSODP-FIR and a sKF: MSE (blue) and relative number of transmitted events (orange) per time step.

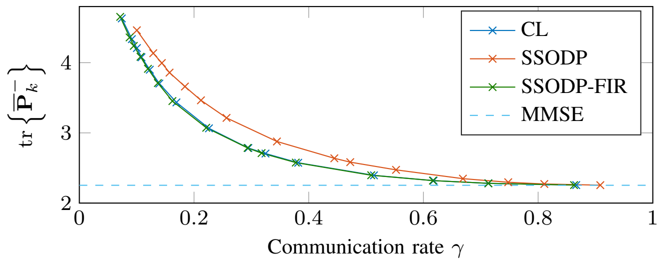


Fig. 4: Error covariance over communication rate for the oscillating stable system using CL and SSODP-FIR.

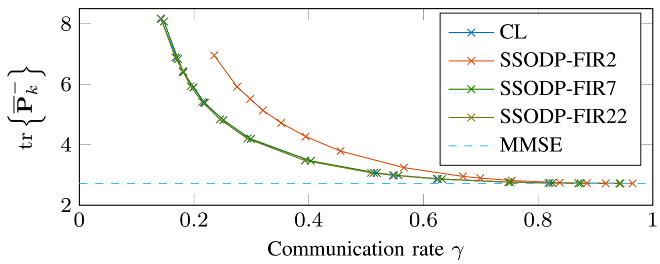


Fig. 5: Error covariance over communication Rate for the unstable system using CL, SSODP and SSODP-FIR.

correspond to the bounds of $\overline{\mathbf{P}}_k^-$, where the upper one is determined using (14), the lower one is the MMSE solution from (13), which is reached by periodic transmission using a standard Kalman filter. As expected, all schemes approach this bound for $\gamma = 1$.

In Figure 2, the communication rate is shown over the choice of $\mathbf{Z}$. The simulated rates are bounded by the theoretically derived bounds. Hereby, the KF bounds (eq. (13) and (14)) correspond to the closed-loop case. Due to the short horizon m of the FIR estimator, the FIR communication bounds are higher than the KF bounds. For $m \to \infty$, the FIR bounds (eq. (15) and (16)) would approach the KF bounds.

B. Initialization and Temporary Errors

Further benefits of implementing an SSODP-FIR scheme at the sensor can, in particular, be seen when erroneous initial conditions have been chosen or unmodeled errors occur. Fig. 3 compares the SSODP-FIR scheme with a Kalman filter-based trigger, which in this case corresponds to the CL scheme. $\mathbf{Z}$ is set to 7. The SSODP-FIR scheme is less susceptible to both erroneous initial conditions and outliers. The transmission rate under the SSODP-FIR scheme decreases fast while the Kalman filter-based trigger bursts transmissions for a far longer period. The reaction time to such errors has a severe impact on the performance of the remote estimator. As a result, the MSE of the remote estimator can recover faster under the SSODP-FIR scheme, which suffers from such unmodeled errors only over a short, finite horizon. A limited-memory trigger design can, hence, prove effective to increase robustness and performance of event-based state estimation.

C. Oscillating Stable System

A more challenging example is the following oscillating second-order system:

$$\mathbf{A} = \begin{bmatrix} -0.85 & -0.35 \\ 0.35 & -0.85 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 0.01 & 0.1 \\ 0.1 & 1 \end{bmatrix},$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{R} = 0.1,$$

which is shown in Fig. 4. In this case, SSODP-FIR with $m = 4$ significantly outperforms the SSODP scheme. SSODP only partially accounts for the highly dynamic system behaviour by using the system model but assuming the system to be in a steady state, which is only approximately true for finite non-triggering intervals. Despite the short horizon $m = 4$, SSODP-FIR and CL again show comparable performance.

D. Unstable System

The third example is given by an unstable system:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.3 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 1 & 0.7 \\ 0.7 & 0.49 \end{bmatrix},$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{R} = 1.$$

Since SSODP cannot be used for unstable systems, only CL and SSODP-FIR are shown in Fig. 5. In this example, multiple horizon lengths m are compared and presented alongside CL, which acts as a lower bound. It can be seen that the estimation quality increases quite a lot between $m = 2$ and $m = 7$, but only very little from $m = 7$ to $m = 22$. This shows how fast the error covariance $\mathbf{E}$ approaches its final value and how little effort has to be put into calculating a state estimate which is good enough to enable efficient triggering and consequently efficient event-based estimation.

V. CONCLUSIONS

As can be seen from the simulation results, the novel FIR-based stochastic trigger can combine the simplicity of the SSODP scheme and the efficiency of the CL scheme while overcoming their deficiencies. SSODP-FIR does not pose any requirements on the system behaviour besides detectability of the pair $(\mathbf{A}, \mathbf{C})$, especially, the system does not have to be stable. Furthermore, no bidirectional communication between the sensor and the receiver as for CL is needed. A representative state estimate $\tilde{\underline{\mathbf{x}}}_k^{\mathrm{FIR}}$ is calculated at the sensor and shared with the receiver when an event is triggered. The exploitation of simultaneous information at the sensor and receiver leads to an efficient triggering policy only triggering events if the system behavior deviates from the system model. An optimal state estimate at the sensor could be obtained using a simultaneous KF; however, SSODP-FIR shows to be advantageous over sKF as well: Due to its finite time horizon, the FIR estimate is less susceptible to outliers and erroneous initializations. The MSE and the transmission rate recover faster than for sKF or similarly CL. Regarding computational load, SSODP-FIR performs well in any time-invariant scenario as the pseudo-inverse required for the least squares algorithm can be precomputed, and no matrix inversion as necessary for the computation of the Kalman gain in sKF is involved.

Tasks for future research are the development of a Colored Noise KF to reduce the communication effort even further by not sending the current measurement to the sensor and instead using $\underline{\tilde{\mathbf{x}}}_k^{\mathrm{FIR}}$. Moreover, further investigations towards finding tighter bounds or an analytic expression for the communication rate should be conducted. Particular focus is to be directed towards imperfect communication links [23], where package loss or delays occur that impede the use of implicit information. In this case, stochastic triggers generally do not preserve Gaussianity [24], and solutions to model the implicit measurement information need to be found.

APPENDIX

A. Choice of $\underline{\mathbf{c}}_k$

It is shown that any $\underline{\mathbf{c}}_k$ that is conditionally independent of $\underline{\mathbf{y}}_k$ given the state $\underline{\mathbf{x}}_k$ can be chosen in (10). For $\gamma_k = 1$, this is obviously the case since $\underline{\mathbf{c}}_k$ cancels

$$\begin{aligned}\hat{\underline{\mathbf{x}}}_k &= \hat{\underline{\mathbf{x}}}_k^- + \mathbf{K}_k\left(\gamma_k\,\underline{\mathbf{z}}_k - \hat{\underline{\mathbf{z}}}_k^-\right)\\ &= \hat{\underline{\mathbf{x}}}_k^- + \mathbf{K}_k\left(\underline{\mathbf{y}}_k - \underline{\mathbf{c}}_k - \mathbf{C}\,\hat{\underline{\mathbf{x}}}_k^- + \underline{\mathbf{c}}_k\right)\\ &= \hat{\underline{\mathbf{x}}}_k^- + \mathbf{K}_k\left(\underline{\mathbf{y}}_k - \mathbf{C}\hat{\underline{\mathbf{x}}}_k^-\right).\end{aligned}$$

For $\gamma_k = 0$, further investigations are necessary:

$$\begin{aligned}&\Pr\left\{\gamma_k = 0\,\middle|\,\underline{\mathbf{x}}_k\right\} = \int_{-\infty}^{\infty}\Pr\left\{\gamma_k = 0, \underline{\mathbf{y}}_k\,\middle|\,\underline{\mathbf{x}}_k\right\}\mathrm{d}\,\underline{\mathbf{y}}_k\\ &= \int_{-\infty}^{\infty}\Pr\left\{\gamma_k = 0\,\middle|\,\underline{\mathbf{y}}_k, \underline{\mathbf{x}}_k\right\}\cdot\Pr\left\{\underline{\mathbf{y}}_k\,\middle|\,\underline{\mathbf{x}}_k\right\}\mathrm{d}\,\underline{\mathbf{y}}_k\\ &= a\cdot\int_{-\infty}^{\infty}\exp\left(-0.5(\underline{\mathbf{y}}_k - \underline{\mathbf{c}}_k)^{\mathrm{T}}\,\mathbf{Z}^{-1}\left(\underline{\mathbf{y}}_k - \underline{\mathbf{c}}_k\right)\right)\cdot\\ &\qquad \exp\left(-0.5(\underline{\mathbf{y}}_k - \mathbf{C}\,\underline{\mathbf{x}}_k)^{\mathrm{T}}\,\mathbf{R}^{-1}\left(\underline{\mathbf{y}}_k - \mathbf{C}\,\underline{\mathbf{x}}_k\right)\right)\mathrm{d}\,\underline{\mathbf{y}}_k\\ &= a\cdot\exp\left(-0.5(\underline{\mathbf{c}}_k - \mathbf{C}\,\underline{\mathbf{x}}_k)^{\mathrm{T}}(\mathbf{Z}+\mathbf{R})^{-1}(\underline{\mathbf{c}}_k - \mathbf{C}\,\underline{\mathbf{x}}_k)\right)\,,\end{aligned}$$

with constant terms being summarized in a.

Hence, this likelihood encodes the implicit measurement $\underline{\mathbf{c}}_k$ with error covariance $\mathbf{Z} + \mathbf{R}$ and consequently leads to the measurement update in (9) and (11).

REFERENCES

[1] G. Battistelli, A. Benavoli, and L. Chisci, “Data-driven Communication for State Estimation with Sensor Networks,” *Automatica*, vol. 48, no. 5, pp. 926–935, May 2012.

[2] J. Sijs and M. Lazar, “Event Based State Estimation with Time Synchronous Updates,” *IEEE Trans. Autom. Control*, vol. 57, no. 10, pp. 2650–2655, Oct. 2012.

[3] D. Shi, L. Shi, and T. Chen, *Event-Based State Estimation: A Stochastic Perspective*, ser. Studies in Systems, Decision and Control. Cham: Springer International Publishing, 2016, vol. 41.

[4] S. Trimpe and R. D’Andrea, “Event-Based State Estimation with Variance-Based Triggering,” *IEEE Trans. Autom. Control*, vol. 59, no. 12, pp. 3266–3281, Dec. 2014.

[5] J. Sijs and M. Lazar, “On Event Based State Estimation,” in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, R. Majumdar and P. Tabuada, Eds. Springer Berlin Heidelberg, Jan. 2009, no. 5469, pp. 336–350.

[6] J. Wu, Q. S. Jia, K. H. Johansson, and L. Shi, “Event-Based Sensor Data Scheduling: Trade-Off Between Communication Rate and Estimation Quality,” *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 1041–1046, Apr. 2013.

[7] A. Molin, H. Sandberg, and K. H. Johansson, “Consistency-Preserving Event-Triggered Estimation in Sensor Networks,” in *Proc. 54th IEEE Conf. Decision and Control (CDC 2015)*, Osaka, Japan, Dec. 2015, pp. 7494–7501.

[8] S. Trimpe, “Stability Analysis of Distributed Event-Based State Estimation,” in *Proc. 53rd IEEE Conf. Decision and Control (CDC 2014)*, Los Angeles, CA, USA, Dec. 2014.

[9] X. Bian, X. R. Li, and V. P. Jilkov, “Remote State Estimation with Data-driven Communication and Guaranteed Stability,” in *Proc. 21st Int. Conf. Inform. Fusion (Fusion 2018)*, Cambridge, UK, Jul. 2018.

[10] J. Sijs, L. Kester, and B. Noack, “A Study on Event Triggering Criteria for Estimation,” in *Proc. 17th Int. Conf. Inform. Fusion (Fusion 2014)*, Salamanca, Spain, Jul. 2014.

[11] J. Sijs, B. Noack, and U. D. Hanebeck, “Event-based State Estimation with Negative Information,” in *Proc. 16th Int. Conf. Inform. Fusion (Fusion 2013)*, Istanbul, Turkey, Jul. 2013.

[12] J. Sijs, B. Noack, M. Lazar, and U. D. Hanebeck, in *Event-Based Control and Signal Processing*, M. Miskowicz, Ed. CRC Press, Nov. 2015, ch. Time-Periodic State Estimation with Event-Based Measurement Updates, pp. 261–279.

[13] D. Shi, T. Chen, and L. Shi, “An Event-triggered Approach to State Estimation With Multiple Point- and Set-valued Measurements,” *Automatica*, vol. 50, no. 6, pp. 1641–1648, Jun. 2014.

[14] B. Noack, F. Pfaff, and U. D. Hanebeck, “Optimal Kalman Gains for Combined Stochastic and Set-Membership State Estimation,” in *Proceedings of the 51st IEEE Conference on Decision and Control (CDC 2012)*, Maui, Hawaii, USA, Dec. 2012.

[15] K. Dormann, B. Noack, and U. D. Hanebeck, “Optimally Distributed Kalman Filtering with Data-Driven Communication,” *Sensors*, vol. 18, no. 4, Apr. 2018.

[16] G. Battistelli, L. Chisci, and D. Selvi, “A Distributed Kalman Filter with Event-Triggered Communication and Guaranteed Stability,” *Automatica*, vol. 93, pp. 75–82, Jul. 2018.

[17] D. Han, Y. Mo, J. Wu, S. Weerakkody, B. Sinopoli, and L. Shi, “Stochastic Event-Triggered Sensor Schedule for Remote State Estimation,” *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2661–2675, Oct. 2015.

[18] S. Weerakkody, Y. Mo, B. Sinopoli, D. Han, and L. Shi, “Multi-Sensor Scheduling for State Estimation With Event-Based, Stochastic Triggers,” *IEEE Trans. Autom. Control*, vol. 61, no. 9, pp. 2695–2701, Sep. 2016.

[19] A. Mohammadi and K. N. Plataniotis, “Event-Based Estimation With Information-Based Triggering and Adaptive Update,” *IEEE Trans. Signal Processing*, vol. 65, no. 18, pp. 4924–4939, Sep. 2017.

[20] M. T. Andrén and A. Cervin, “Event-Based State Estimation Using an Improved Stochastic Send-on-Delta Sampling Scheme,” in *Proc. 2016 2nd Int. Conf. on Event-based Control, Communication, and Signal Processing (EBCCSP 2016)*, Krakow, Poland, Jun. 2016.

[21] J. Wu, X. Ren, D. Han, D. Shi, and L. Shi, “Finite-Horizon Gaussianity-Preserving Event-Based Sensor Scheduling in Kalman Filter Applications,” *Automatica*, vol. 72, pp. 100–107, Oct. 2016.

[22] Y. S. Shmaliy, S. Zhao, and C. K. Ahn, “Unbiased Finite Impluse Response Filtering: An Iterative Alternative to Kalman Filtering Ignoring Noise and Initial Conditions,” *IEEE Control Systems*, vol. 37, no. 5, pp. 70–89, Oct. 2017.

[23] J. Sijs and B. Noack, “Event-based State Estimation in a Feedback Loop with Imperfect Communication Links,” in *Proc. 20th IFAC World Congr. (IFAC 2017)*, Toulouse, France, Jul. 2017.

[24] E. Kung, J. Wu, D. Shi, and L. Shi, “On the Nonexistence of Event-Based Triggers That Preserve Gaussian State in Presence of Package-Drop,” in *Proc. 2017 Amer. Control Conf. (ACC 2017)*, Seattle, WA, USA, May 2017.