State Estimation for Ellipsoidally Constrained Dynamic Systems with Set-membership Pseudo Measurements

Benjamin Noack, Marcus Baum, and Uwe D. Hanebeck

Abstract— **In many dynamic systems, the evolution of the state is subject to specific constraints. In general, constraints cannot easily be integrated into the prediction-correction structure of the Kalman filter algorithm. Linear equality constraints are an exception to this rule and have been widely used and studied as they allow for simple closed-form expressions. A common approach is to reformulate equality constraints into pseudo measurements of the state to be estimated. However, equality constraints define deterministic relationships between state components which is an undesirable property in Kalman filtering as this leads to singular covariance matrices. A second problem relates to the knowledge required to identify and define precise constraints, which are met by the system state. In this article, ellipsoidal constraints are introduced that can be employed to model a bounded region, to which the system state is constrained. This concept constitutes an easy-to-use relaxation of equality constraints. In order to integrate ellipsoidal constraints into the Kalman filter structure, a generalized filter framework is utilized that relies on a combined stochastic and set-membership uncertainty representation.**

I. INTRODUCTION

For the problem of estimating the state of a linear dynamic system with white process and measurement noise, the Kalman filter [1] is shown to be optimal in terms of minimum mean squared error and allows for closed-form expressions in the filtering equations. The linear system model that characterizes the evolution of the state takes into account dependencies and relations between the state components. However, in many dynamic system, the state is subject to additional constraints that are not explicitly incorporated in the process model. The motion model of a ground vehicle [2] is an important example, where roads can be treated as additional constraints on the state. Other examples are geometric constraints [3], applications in fault detection [4], and industrial process monitoring [5].

In particular, equality constraints are widely used as they can easily be integrated into the structure of the Kalman filter [6]–[8]. The results of a standard Kalman filter can be projected onto the constraint, which does not alter its prediction-correction cycle. With the concept of pseudo measurements [8], the Kalman filter is extended by a second filtering step in which the equality constraint serves as a perfect, error-free measurement of the state. Perfect measurement information leads to singular covariance matrices, which is an undesirable property in using the Kalman filter. This problem is discussed in [9], and different possibilities of incorporating pseudo measurements are named. A different approach to equality-constrained filtering is to adapt the process and sensor models. Reduced-order Kalman filters and null-space methods are investigated in [10], where the relations defined by the constraint are exploited to reduce the dimensionality of the state space. In particular, an unconstrained subsystem can be identified [11].

In many applications, it is difficult for the user to precisely define equality constraints, and weaker definitions are preferable. However, Kalman filtering becomes more involved if non-equality constraints are to be considered. Inequality constraints are studied in [12], where quadratic programming routines are required to compute an estimate. A different approach to inequality constraints consists of truncating Gaussian densities [13], which is complicated in multidimensional estimation problems. In [14], interval constraints are incorporated into the unscented Kalman filter by adapting the corresponding sigma points.

In this article, ellipsoidal constraints are introduced and investigated that can be employed to define bounded regions, to which the state is constrained. They represent a straightforward generalization of equality constraints. An ellipsoidal constraint is considered as a pseudo measurement of the state that is affected by a set-membership noise term. In order to integrate set-membership pseudo measurements into the Kalman filter structure, the generalized filter framework proposed in [15] and [16] is utilized that relies on a combined stochastic and set-membership uncertainty representation.

II. PROBLEM FORMULATION

Throughout this paper, the following notations and conventions are used. Real-valued vectors are denoted as underlined variables $\underline{x}$, and boldface, lowercase letters $\underline{\mathbf{x}}$ represent random quantities. Matrices are written in uppercase boldface letters $\mathbf{C}$. The matrices $\mathbf{C}^{-1}$ and $\mathbf{C}^{\mathrm{T}}$ are the inverse and transpose, respectively. The vector $\hat{\underline{x}}$ is used for the mean of a random variable, an estimate of an uncertain quantity, or an observation. The matrix $\mathbf{I}$ is the identity matrix of appropriate dimension. An ellipsoid with center $\hat{\underline{c}}$ and shape matrix $\mathbf{X}$ is defined by $\mathcal{E}(\hat{\underline{c}}, \mathbf{X}) = \left\{\underline{x} \in \mathbb{R}^n \,|\, (\hat{\underline{c}} - \underline{x})^{\mathrm{T}}\mathbf{X}^{-1}(\hat{\underline{c}} - \underline{x}) \leq 1\right\}$. An element of $\mathcal{E}(\hat{\underline{c}}, \mathbf{X})$ is denoted by $\underline{\mathfrak{c}}$.

A. Process Model

A linear stochastic dynamic system is considered that is characterized by the discrete-time model

$$\underline{\mathbf{x}}_{k+1} = \mathbf{A}_k \, \underline{\mathbf{x}}_k + \mathbf{B}_k \, \underline{u}_k + \underline{\mathbf{w}}_k, \quad k = 0, 1, \dots \,, \tag{1}$$

This work was supported by the German Research Foundation (DFG).
Benjamin Noack, Marcus Baum, and Uwe D. Hanebeck are with the Intelligent Sensor-Actuator-Systems Laboratory (ISAS), Institute for Anthropomatics and Robotics, Karlsruhe Institute of Technology (KIT), Germany. benjamin.noack@ieee.org, marcus.baum@ieee.org, uwe.hanebeck@ieee.org

where $\mathbf{A}_k \in \mathbb{R}^{n_x \times n_x}$ is the system matrix and $\underline{\mathbf{w}}_k$ is a zero-mean white noise with covariance matrix $\mathrm{Cov}(\underline{\mathbf{w}}_k) = \mathbf{C}_k^{\mathbf{w}} \in \mathbb{R}^{n_x \times n_x}$. An input $\hat{\underline{u}}_k \in \mathbb{R}^{n_u}$ can affect the system with control-input matrix $\mathbf{B}_k \in \mathbb{R}^{n_x \times n_u}$. The initial state has the mean $\mathrm{E}[\underline{\mathbf{x}}_0] = \hat{\underline{x}}_0$ and covariance matrix $\mathrm{Cov}(\underline{\mathbf{x}}_0) = \mathbf{C}_0$.

B. Physical Sensor Model

A sensor measurement $\hat{\underline{z}}_k \in \mathbb{R}^{n_z}$ of the system is related to the state through a linear stochastic model

$$\underline{\mathbf{z}}_k = \mathbf{H}_k \, \underline{\mathbf{x}}_k + \underline{\mathbf{v}}_k, \quad k = 0, 1, \ldots, \tag{2}$$

with zero-mean white noise $\underline{\mathbf{v}}_k$. It has the covariance matrix $\mathrm{Cov}(\underline{\mathbf{v}}_k) = \mathbf{C}_k^{\mathbf{v}} \in \mathbb{R}^{n_z \times n_z}$, and $\mathbf{H}_k \in \mathbb{R}^{n_z \times n_x}$ is the measurement matrix. The vectors $\{\underline{\mathbf{x}}_k\}_{k=0,1,\ldots}$, $\{\underline{\mathbf{w}}_k\}_{k=0,1,\ldots}$, and $\{\underline{\mathbf{v}}_k\}_{k=0,1,\ldots}$ are assumed to be mutually uncorrelated.

C. Constraints on the State

The system state can be subject to different types of constraints. In this paper, the following three constraints are investigated.

Equality Constraints: An equality constraint is defined as

$$\mathbf{D}_k \underline{x}_k = \underline{d}_k \ , \tag{3}$$

where the matrix $\mathbf{D}_k \in \mathbb{R}^{n_d \times n_x}$ and the vector $\underline{d}_k \in \mathbb{R}^{n_d}$ are known, possibly time-varying parameters. In existing work on equality constraints, it is often assumed that $\mathbf{D}_k$ has full row rank. As discussed in [9], this assumption is not required. However, a reasonable requirement is that $\mathrm{rank}(\mathbf{D}_k) < n_x$. Otherwise, the state would be fully defined by the constraint.

Inequality Constraints: With the same parameters $\mathbf{D}_k \in \mathbb{R}^{n_d \times n_x}$ and $\underline{d}_k \in \mathbb{R}^{n_d}$, an inequality constraint

$$\mathbf{D}_k \underline{x}_k \leq \underline{d}_k \ , \tag{4}$$

can be defined which characterizes a convex polytope. Although these constraints appear to be more useful from a practical point of view, they gain less attention in literature than equality constraints as the resulting filtering algorithms are more complicated and may involve quadratic programming procedures.

Ellipsoidal Constraints: In this paper, we introduce and study ellipsoidal constraints given by

$$\mathbf{D}_k \underline{x}_k \in \mathcal{E}(\underline{d}_k, \mathbf{X}_k^{\mathrm{d}}) \tag{5}$$

where $\underline{d}_k \in \mathbb{R}^{n_d}$ represents the midpoint of an ellipsoid and the positive definite matrix $\mathbf{X}_k^{\mathrm{d}} \in \mathbb{R}^{n_d \times n_d}$ defines the shape of the ellipsoid. This constraint displays a simple relaxation of (3); instead of requiring the state to be equal to $\underline{d}_k$, it is assumed to lie in a bounded region around $\underline{d}_k$. An ellipsoidal constraint essentially embodies a quadratic constraint of the form

$$(\underline{d}_k - \mathbf{D}_k \underline{x})^{\mathrm{T}} (\mathbf{X}_k^{\mathrm{d}})^{-1} (\underline{d}_k - \mathbf{D}_k \underline{x}) - 1 \leq 0 \ . \tag{6}$$

In this paper, it is demonstrated that ellipsoidal constraints can easily be incorporated into the state estimation process.

III. UNCONSTRAINED LINEAR ESTIMATION

In this section, two concepts for unconstrained linear estimation are revisited. After considering the prediction-correction cycle of the well-known Kalman filter, its generalization to combined stochastic and set-membership uncertainties is presented. This generalization lays the groundwork for the treatment of ellipsoidal constraints.

A. Kalman Filtering

The Kalman filter has been introduced in [1] and provides an estimate $\hat{\underline{x}}_k^{\mathrm{e}}$ of the state $\underline{x}_k$ such that the mean squared error is minimized, which corresponds to minimizing the trace of the error covariance matrix $\mathbf{C} = \mathrm{E}[\tilde{\underline{\mathbf{x}}}_k \tilde{\underline{\mathbf{x}}}_k^{\mathrm{T}}]$ with

$$\tilde{\underline{\mathbf{x}}}_k = \hat{\underline{x}}_k^{\mathrm{e}} - \underline{\mathbf{x}}_k \ .$$

The state estimate is dynamically computed in prediction and update steps. The estimator is typically initialized with a prior estimate $\hat{\underline{x}}_0^{\mathrm{e}}$ with covariance matrix $\mathbf{C}_0^{\mathrm{e}}$.

Prediction step: The prediction step of the Kalman filter employs the system model (1) to compute the current state estimate according to

$$\hat{\underline{x}}_{k+1}^{\mathrm{p}} := \mathrm{E}[\underline{\mathbf{x}}_{k+1} \,|\, \hat{\underline{z}}_{0:k}] = \mathbf{A}_k \, \hat{\underline{x}}_k^{\mathrm{e}} + \mathbf{B}_k \, \hat{\underline{u}}_k \tag{7a}$$

and

$$\mathbf{C}_{k+1}^{\mathrm{p}} = \mathrm{E}[(\tilde{\underline{\mathbf{x}}}_{k+1})(\tilde{\underline{\mathbf{x}}}_{k+1})^{\mathrm{T}}] = \mathbf{A}_k \mathbf{C}_k^{\mathrm{e}} \mathbf{A}_k^{\mathrm{T}} + \mathbf{C}_k^{\mathbf{w}} \tag{7b}$$

for the conditional mean and the error covariance matrix, respectively.

Update step: In the correction or filtering step, measurements of the state are incorporated to update the estimate. The measurement are related to the state by (2). By means of the Kalman gain

$$\mathbf{K}_k = \mathbf{C}_k^{\mathrm{p}} \mathbf{H}_k^{\mathrm{T}} (\mathbf{C}_k^{\mathbf{v}} + \mathbf{H}_k \mathbf{C}_k^{\mathrm{p}} \mathbf{H}_k^{\mathrm{T}})^{-1} \ , \tag{8}$$

the combination

$$\begin{aligned} \hat{\underline{x}}_k^{\mathrm{e}} &:= \mathrm{E}[\underline{\mathbf{x}}_k \,|\, \hat{\underline{z}}_{0:k}] = \hat{\underline{x}}_k^{\mathrm{p}} + \mathbf{K}_k (\hat{\underline{z}}_k - \mathbf{H}_k \hat{\underline{x}}_k^{\mathrm{p}}) \\ &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \hat{\underline{x}}_k^{\mathrm{p}} + \mathbf{K}_k \, \hat{\underline{z}}_k \end{aligned} \tag{9a}$$

can be computed, which minimizes the trace of the error covariance matrix

$$\begin{aligned} \mathbf{C}_k^{\mathrm{e}} &= \mathrm{E}[(\tilde{\underline{\mathbf{x}}}_k)(\tilde{\underline{\mathbf{x}}}_k)^{\mathrm{T}}] = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{C}_k^{\mathrm{p}} \\ &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{C}_k^{\mathrm{p}} (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^{\mathrm{T}} + \mathbf{K}_k \mathbf{C}_k^{\mathbf{v}} \mathbf{K}_k^{\mathrm{T}} \ , \end{aligned} \tag{9b}$$

where the latter formula is the Joseph form.

B. Combined Stochastic and Set-membership Filtering

In [15] and [16], a concept for the simultaneous treatment of stochastic and set-membership errors has been proposed. It has been derived for processes and sensor systems that are affected by possibly non-stochastic errors. More precisely, these errors are assumed to be unknown but bounded. In the system model (1), an additional, additive error term $\underline{\mathrm{w}}_k \in \mathcal{E}(\underline{0}, \mathbf{X}_k^{\mathrm{w}})$ is considered. In the sensor model (2), measurements can be additionally affected by an unknown but bounded noise $\underline{\mathrm{v}}_k \in \mathcal{E}(\underline{0}, \mathbf{X}_k^{\mathrm{v}})$. Both error terms are characterized by ellipsoids with shape matrices $\mathbf{X}_k^{\mathrm{w}} \in \mathbb{R}^{n_x \times n_x}$ and $\mathbf{X}_k^{\mathrm{v}} \in \mathbb{R}^{n_z \times n_z}$.

Due to the presence of two different types of uncertainties, the estimation error is also decomposed into a stochastic and a set-membership error term according to

$$\tilde{\underline{\mathbf{x}}}_k + \underline{\mathbb{x}}_k = \hat{\underline{x}}_k^{\mathrm{e}} - \underline{\mathbf{x}}_k \ ,$$

which means that the estimation uncertainty is represented by a random deviation $\tilde{\underline{\mathbf{x}}}_k \sim \mathcal{N}(\underline{0}, \mathbf{C}_k^{\mathrm{e}})$ and an unknown but bounded term $\underline{\mathbb{x}}_k \in \mathcal{E}(\underline{0}, \mathbf{X}_k^{\mathrm{e}})$. An estimate $\hat{\underline{x}}_k^{\mathrm{e}}$ is consequently associated to two uncertainty characteristics, i.e., an error covariance matrix $\mathbf{C}_k^{\mathrm{e}}$ and an ellipsoidal shape matrix $\mathbf{X}_k^{\mathrm{e}}$. The mean squared error is then given by

$$\begin{aligned}\mathrm{E}\left[(\hat{\underline{x}}_k^{\mathrm{e}} - \underline{\mathbf{x}}_k)^{\mathrm{T}}(\hat{\underline{x}}_k^{\mathrm{e}} - \underline{\mathbf{x}}_k)\right] &= \underbrace{\mathrm{E}\left[(\tilde{\underline{\mathbf{x}}}_k)^{\mathrm{T}}(\tilde{\underline{\mathbf{x}}}_k)\right]}_{=\,\mathrm{trace}(\mathbf{C}^{\mathrm{e}})} + \underbrace{(\underline{\mathbb{x}}_k)^{\mathrm{T}}(\underline{\mathbb{x}}_k)}_{\leq\,\mathrm{trace}(\mathbf{X}^{\mathrm{e}})} \\ &\leq \mathrm{trace}(\mathbf{C}_k^{\mathrm{e}} + \mathbf{X}_k^{\mathrm{e}}) \ . \end{aligned} \tag{10}$$

In the presence of set-membership uncertainties, the objective consists of minimizing the maximum possible mean squared error, which is $\mathrm{trace}(\mathbf{C}_k^{\mathrm{e}} + \mathbf{X}_k^{\mathrm{e}})$.

Prediction step: In the prediction step of the combined filter, mean and covariance matrix are still given by the formulas (7a) and (7b), i.e.,

$$\hat{\underline{x}}_{k+1}^{\mathrm{p}} = \mathbf{A}_k \, \hat{\underline{x}}_k^{\mathrm{e}} + \mathbf{B}_k \, \hat{\underline{u}}_k \tag{11a}$$

and

$$\mathbf{C}_{k+1}^{\mathrm{p}} = \mathbf{A}_k \mathbf{C}_k^{\mathrm{e}} \mathbf{A}_k^{\mathrm{T}} + \mathbf{C}_k^{\mathbf{w}} \ , \tag{11b}$$

respectively. In addition, the shape matrix $\mathbf{X}_{k+1}^{\mathrm{p}}$ related to the predicted set-membership error $\underline{\mathbb{x}}_{k+1} = \mathbf{A}_k \, \underline{\mathbb{x}}_k + \underline{\mathbb{w}}_k$ with $\underline{\mathbb{w}}_k \in \mathcal{E}(\underline{0}, \mathbf{X}_k^{\mathrm{w}})$ has to be computed. For this reason, the Minkoswki sum

$$\underline{\mathbb{x}}_{k+1} \in \mathcal{E}(\underline{0}, \mathbf{A}_k \mathbf{X}_k^{\mathrm{e}} \mathbf{A}_k) \oplus \mathcal{E}(\underline{0}, \mathbf{X}_k^{\mathrm{w}}) \subseteq \mathcal{E}(\underline{0}, \mathbf{X}_{k+1}^{\mathrm{p}})$$

has to be considered. From ellipsoidal calculus [17], it is known that the Minkoswki is tightly bounded by a family of ellipsoids with shape matrices

$$\mathbf{X}_{k+1}^{\mathrm{p}}(\omega) = \tfrac{1}{\omega} \mathbf{A}_k \mathbf{X}_k^{\mathrm{e}} \mathbf{A}_k^{\mathrm{T}} + \tfrac{1}{1-\omega} \mathbf{X}_k^{\mathrm{w}} \tag{11c}$$

and $\omega \in (0,1)$. The parameter ω is typically chosen to minimize the trace of (11c) so as to minimize the error (10). In this case, a simple closed-form solution for determining ω is attainable.

Update step: In the filtering step, a prior estimate $\hat{\underline{x}}_k^{\mathrm{p}}$ with error characteristics $\mathbf{C}_k^{\mathrm{p}}$ and $\mathbf{X}_k^{\mathrm{p}}$ is to be updated with measurement information $\hat{\underline{z}}$. The gain has to incorporate both stochastic and set-membership measurement uncertainties $\underline{\mathbf{v}}_k \sim \mathcal{N}(\underline{0}, \mathbf{C}_k^{\mathrm{v}})$ and $\underline{\mathbb{v}} \in \mathcal{E}(\underline{0}, \mathbf{X}_k^{\mathrm{v}})$ and is given by

$$\begin{aligned}\mathbf{K}_k(\omega) = &\left(\tfrac{1}{\omega} \mathbf{X}_k^{\mathrm{p}} \mathbf{H}_k^{\mathrm{T}} + \mathbf{C}_k^{\mathrm{p}} \mathbf{H}_k^{\mathrm{T}}\right) \cdot \\ &\left(\tfrac{1}{\omega} \mathbf{H}_k \mathbf{X}_k^{\mathrm{p}} \mathbf{H}_k^{\mathrm{T}} + \tfrac{1}{1-\omega} \mathbf{X}_k^{\mathrm{v}} + \mathbf{H}_k \mathbf{C}_k^{\mathrm{p}} \mathbf{H}^{\mathrm{T}} + \mathbf{C}_k^{\mathrm{v}}\right)^{-1} .\end{aligned} \tag{12}$$

Apparently, the gain also depends on a weighting parameter $\omega \in (0,1)$. As in the prediction step, a Minkoswki sum is responsible for this parameter. In line with (9), estimate and covariance matrix are updated according to

$$\hat{\underline{x}}_k^{\mathrm{e}}(\omega) = (\mathbf{I} - \mathbf{K}_k(\omega)\mathbf{H}_k)\hat{\underline{x}}_k^{\mathrm{p}} + \mathbf{K}_k(\omega)\hat{\underline{z}}_k \tag{13a}$$

and

$$\begin{aligned}\mathbf{C}_k^{\mathrm{e}}(\omega) = &(\mathbf{I} - \mathbf{K}_k(\omega)\mathbf{H}_k)\mathbf{C}_k^{\mathrm{p}}(\mathbf{I} - \mathbf{K}_k(\omega)\mathbf{H}_k)^{\mathrm{T}} \\ &+ \mathbf{K}_k(\omega)\mathbf{C}_k^{\mathrm{v}}\mathbf{K}_k(\omega)^{\mathrm{T}} \ ,\end{aligned} \tag{13b}$$

respectively. The updated shape matrix for the set-membership uncertainty becomes

$$\begin{aligned}\mathbf{X}_k^{\mathrm{e}}(\omega) = \ &\tfrac{1}{\omega}(\mathbf{I} - \mathbf{K}_k(\omega)\mathbf{H}_k)\mathbf{X}_k^{\mathrm{p}}(\mathbf{I} - \mathbf{K}_k(\omega)\mathbf{H}_k)^{\mathrm{T}} \\ &+ \tfrac{1}{1-\omega}\mathbf{K}_k(\omega)\mathbf{X}_k^{\mathrm{v}}\mathbf{K}_k(\omega)^{\mathrm{T}} \ .\end{aligned} \tag{13c}$$

Each $\omega \in (0,1)$ is admissible, but the parameter is typically determined to minimize $\mathrm{trace}\left(\mathbf{C}_k^{\mathrm{e}}(\omega) + \mathbf{X}_k^{\mathrm{e}}(\omega)\right)$, which is the bound in (10). A simple bisection method can be used to solve the convex optimization problem for determining the trace-minimal ω.

IV. Linear Equality Constrained Systems

In many dynamic systems, the evolution of the state is subject to specific constraints. Linear equality constraints in the form of (3) are widely used and imply that some state variables are linearly dependent of each other. This deterministic dependency between state components can be exploited to reduce the dimensionality of the state space. Proposed concepts are the reduced-order Kalman filter [10] and the null-space method [18]. In [11], a direct elimination method is employed to identify an unconstrained subsystem while the remaining system variables are deterministically generated from the unconstrained variables.

Important instances of linear equality-constrained estimation principles are based on projections and pseudo measurements [8]. These methods do not require modifications of the considered system dynamics but utilize the constraint (3) to update the state estimate.

A. Projection-based Methods

Projection-based approaches are formulated as a minimization problem

$$\breve{\underline{x}}_k = \arg\min_{\underline{x}_k} (\underline{x}_k - \hat{\underline{x}}_k^{\mathrm{e}})^{\mathrm{T}} \mathbf{W}_k^{-1} (\underline{x}_k - \hat{\underline{x}}_k^{\mathrm{e}}) \ ,$$

such that

$$\mathbf{D}_k \underline{x}_k = \underline{d}_k \ .$$

The solution of this minimization problem has the form

$$\breve{\underline{x}}_k = \hat{\underline{x}}_k - \mathbf{W}_k \mathbf{D}_k^{\mathrm{T}} (\mathbf{D} \mathbf{W}_k \mathbf{D}^{\mathrm{T}})^{-1} (\mathbf{D}_k \hat{\underline{x}}_k - \underline{d}_k) \ . \tag{14}$$

In [8], different approaches have been discussed that all lead to the parameterization (14). The covariance matrix $\mathbf{C}_k^{\mathrm{e}}$ in (9b) can be considered to be the optimal choice of the weighting matrix $\mathbf{W}_k$ as it minimizes the resulting error covariance matrix $\breve{\mathbf{C}} = \mathrm{E}[(\breve{\underline{x}} - \underline{\mathbf{x}})(\breve{\underline{x}} - \underline{\mathbf{x}})^{\mathrm{T}}]$. The projection is applied to each estimate provided by the underlying Kalman filter.

B. Pseudo Measurements

Besides a projection-based approach, the constraint can be regarded as an error-free measurement equation

$$\breve{\underline{z}}_k = \breve{\mathbf{H}}_k \underline{x}_k \tag{15}$$

that generates the pseudo measurement $\breve{\underline{z}}_k = \underline{d}_k$ and has the measurement matrix $\breve{\mathbf{H}}_k = \mathbf{D}_k$. In [9], different means of incorporating pseudo measurements into the Kalman filter have been discussed. A sequential processing consists of two subsequent filtering steps. The first step is the Kalman update (9), where the standard Kalman gain (8) is used to incorporate the physical measurement (2). The second filtering step utilizes the error-free pseudo measurement (15), and the corresponding gain becomes

$$\breve{\mathbf{K}}_k = \mathbf{C}_k^{\mathrm{e}} \breve{\mathbf{H}}_k^{\mathrm{T}} \left(\breve{\mathbf{H}}_k \mathbf{C}_k^{\mathrm{e}} \breve{\mathbf{H}}_k^{\mathrm{T}} \right)^{-1} ,$$

which is utilized to update the state estimate according to (9), i.e.,

$$\breve{\underline{x}}_k^{\mathrm{e}} = \hat{\underline{x}}_k^{\mathrm{e}} + \breve{\mathbf{K}}_k (\breve{\underline{z}}_k - \breve{\mathbf{H}}_k \hat{\underline{x}}_k^{\mathrm{e}}) \tag{16}$$

and

$$\breve{\mathbf{C}}_k^{\mathrm{e}} = (\mathbf{I} - \breve{\mathbf{K}}_k \breve{\mathbf{H}}_k) \mathbf{C}_k^{\mathrm{e}} \ . \tag{17}$$

The ordering of the filtering steps can be reversed, and the pseudo measurement can be incorporated before the physical measurement is exploited. A third possibility is a batch processing of both physical and pseudo measurements by means of the combined measurement equation

$$\begin{bmatrix} \underline{\mathbf{z}}_k \\ \breve{\underline{z}}_k \end{bmatrix} = \begin{bmatrix} \mathbf{H}_k \\ \breve{\mathbf{H}}_k \end{bmatrix} \underline{\mathbf{x}}_k + \begin{bmatrix} \underline{\mathbf{v}}_k \\ \underline{0} \end{bmatrix} , \tag{18}$$

which is then used in the Kalman filter formulas. The combined measurement error $\bar{\underline{\mathbf{v}}}_k = \left[\begin{smallmatrix} \underline{\mathbf{v}}_k \\ \underline{0} \end{smallmatrix}\right]$ now has the singular covariance matrix $\mathrm{Cov}(\bar{\underline{\mathbf{v}}}_k) = \left[\begin{smallmatrix} \mathbf{C}_k^{\mathrm{v}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right]$.

An important point to be emphasized is that (14) and (16) are identical if $\mathbf{W} = \mathbf{C}_k^{\mathrm{e}}$ holds. However, the pseudo-measurement approach also computes an update of the estimate (16) and the covariance matrix (17), which then enter the subsequent prediction-correction cycle of the Kalman filter. Such an update does not take place in the projection method. An undesirable property is the singular covariance matrix (17), which is due to the noise-free model (15) and possibly leads to numerical problems. In general, modeling of constraints deserves careful attention and, in particular, the requirement of equality can often be too restrictive.

V. LINEAR INEQUALITY CONSTRAINED SYSTEMS

In [12] and [13], inequality constraints (4) have been studied. Analogously to Sec. IV, a minimization problem

$$\breve{\underline{x}}_k = \arg\min_{\underline{x}_k} (\underline{x}_k - \hat{\underline{x}}_k)^{\mathrm{T}} \mathbf{W}_k^{-1} (\underline{x}_k - \hat{\underline{x}}_k) \ , \tag{19}$$

subject to the inequality

$$\mathbf{D}_k \underline{x}_k \leq \underline{d}_k \tag{20}$$

has to be solved. Such a quadratic programming problem can be tackled with the aid of active set methods. As explained in [12], for a solution of (19), only a number of $s \leq n_d$ rows of $\mathbf{D}_k$ and components of $\underline{d}_k$ is active, which are denoted by $\bar{\mathbf{D}}_k$ and $\bar{\underline{d}}_k$. If these active elements are known, the constraint (20) becomes a linear equality constraint

$$\bar{\mathbf{D}}_k \underline{x}_k = \bar{\underline{d}}_k$$

and methods from Sec. IV can be applied. However, active set methods are iterative quadratic programming routines and can be too cumbersome.

VI. LINEAR ELLIPSOIDALLY CONSTRAINED SYSTEMS

In this section, ellipsoidal constraints in the form of (5) are investigated, which are equivalent to quadratic constraints (6). By means of these constraints, linear relationships between state components can be modeled to be confined to bounded regions. For instance, the difference between two components can be restricted to a bounded interval.

For the purpose of treating ellipsoidal constraints (5), the concept of pseudo measurements (15) is generalized to a set-membership model

$$\breve{\underline{z}}_k = \breve{\mathbf{H}}_k \underline{x}_k + \underline{\mathbb{d}}_k \ , \quad k = 0, 1, \ldots \tag{21}$$

with $\breve{\underline{z}}_k = \underline{d}_k$ and $\breve{\mathbf{H}}_k = \mathbf{D}_k$, where $\underline{\mathbb{d}}_k$ is a set-membership error term bounded by the ellipsoid $\mathcal{E}(\underline{0}, \mathbf{X}_k^{\mathrm{d}})$. Consequently, physical measurements are related to a purely stochastic sensor model (2), and pseudo measurements are generated by a purely set-membership model (21). Due to the hybrid structure of the involved measurement equations, the generalized Kalman filter explained in Sec. III-B is predestined for this estimation problem.

A. Ellipsoidally Constrained Kalman Filtering

The generalized Kalman filter can be still initialized by a prior estimate $\hat{\underline{x}}_0^{\mathrm{e}}$ with covariance matrix $\mathbf{C}_0^{\mathrm{e}}$ as it is done for the standard Kalman filter. Set-membership error characteristics $\mathbf{X}_k^{\mathrm{e}}$ come into play as soon as the ellipsoidal constraint (5) is applied and exploited. This matrix can therefore be initialized with $\mathbf{X}_0^{\mathrm{e}} = \mathbf{0}$.

Prediction Step: In order to compute a prior estimate for the subsequent time step $k+1$, the prediction formulas (11) can be applied. The formula for the ellipsoidal shape matrix conveniently reduces to

$$\mathbf{X}_{k+1}^{\mathrm{p}} = \mathbf{A}_k \mathbf{X}_k^{\mathrm{e}} \mathbf{A}_k^{\mathrm{T}}$$

as a purely stochastic system model (1) is considered and no additional unknown but bounded noise is present.

Update Step: The measurement update is subdivided into two substeps. In the first substep, physical measurement information $\hat{\underline{z}}_k$ related to the model (2) is incorporated. Due to the absence of unknown but bounded measurement noise, the gain (12) can be simplified to

$$\mathbf{K}_k = \left(\mathbf{X}_k^{\mathrm{p}} \mathbf{H}_k^{\mathrm{T}} + \mathbf{C}_k^{\mathrm{p}} \mathbf{H}_k^{\mathrm{T}} \right) \left(\mathbf{H}_k \mathbf{X}_k^{\mathrm{p}} \mathbf{H}_k^{\mathrm{T}} + \mathbf{H}_k \mathbf{C}_k^{\mathrm{p}} \mathbf{H}^{\mathrm{T}} + \mathbf{C}_k^{\mathrm{v}} \right)^{-1} .$$

The update formulas (13) become

$$\hat{\underline{x}}_k^{\mathrm{e}} = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\hat{\underline{x}}_k^{\mathrm{p}} + \mathbf{K}_k \hat{\underline{z}}_k \ , \tag{22a}$$

$$\mathbf{C}_k^{\mathrm{e}} = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{C}_k^{\mathrm{p}}(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^{\mathrm{T}} + \mathbf{K}_k \mathbf{C}_k^{\mathrm{v}} \mathbf{K}_k^{\mathrm{T}} \ , \tag{22b}$$

and

$$\mathbf{X}_k^{\mathrm{e}} = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{X}_k^{\mathrm{p}}(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^{\mathrm{T}} \ . \tag{22c}$$

Apparently, the update with physical measurement information does not require a minimization over the parameter ω.

In the second substep, the constraint (5) is taken into account by employing the pseudo measurement related to the model (21). This filtering step is applied to the estimation parameters (22) of the previous, physical update step. With the pseudo measurement noise $\underline{\mathbb{d}} \in \mathcal{E}(\underline{0}, \mathbf{X}_k^{\mathbb{d}})$, the gain (12) yields

$$\breve{\mathbf{K}}_k(\omega) = \left(\tfrac{1}{\omega}\mathbf{X}_k^{\mathrm{e}}\mathbf{H}_k^{\mathrm{T}} + \mathbf{C}_k^{\mathrm{e}}\mathbf{H}_k^{\mathrm{T}}\right) \cdot \left(\tfrac{1}{\omega}\mathbf{H}_k\mathbf{X}_k^{\mathrm{e}}\mathbf{H}_k^{\mathrm{T}} + \tfrac{1}{1-\omega}\mathbf{X}_k^{\mathbb{d}} + \mathbf{H}_k\mathbf{C}_k^{\mathrm{e}}\mathbf{H}^{\mathrm{T}}\right)^{-1} \ .$$

The formulas (13) for the pseudo-measurement update are then given by

$$\breve{\underline{x}}_k^{\mathrm{e}}(\omega) = (\mathbf{I} - \breve{\mathbf{K}}_k(\omega)\breve{\mathbf{H}}_k)\hat{\underline{x}}_k^{\mathrm{e}} + \breve{\mathbf{K}}_k(\omega)\breve{\underline{z}}_k \ , \tag{23a}$$

$$\breve{\mathbf{C}}_k^{\mathrm{e}}(\omega) = (\mathbf{I} - \breve{\mathbf{K}}_k(\omega)\breve{\mathbf{H}}_k)\mathbf{C}_k^{\mathrm{e}}(\mathbf{I} - \breve{\mathbf{K}}_k(\omega)\breve{\mathbf{H}}_k)^{\mathrm{T}} \ , \tag{23b}$$

and

$$\breve{\mathbf{X}}_k^{\mathrm{e}}(\omega) = \tfrac{1}{\omega}(\mathbf{I} - \breve{\mathbf{K}}_k(\omega)\breve{\mathbf{H}}_k)\mathbf{X}_k^{\mathrm{p}}(\mathbf{I} - \breve{\mathbf{K}}_k(\omega)\breve{\mathbf{H}}_k)^{\mathrm{T}} + \tfrac{1}{1-\omega}\breve{\mathbf{K}}_k(\omega)\mathbf{X}_k^{\mathbb{d}}\breve{\mathbf{K}}_k(\omega) \ . \tag{23c}$$

The parameter ω is to be chosen to minimize the bound $\mathrm{trace}\left(\breve{\mathbf{C}}_k^{\mathrm{e}}(\omega) + \breve{\mathbf{X}}_k^{\mathrm{e}}(\omega)\right)$ on the mean squared error. The constrained estimate $\breve{\underline{x}}_k^{\mathrm{e}}$ with covariance and shape matrices $\breve{\mathbf{C}}_k^{\mathrm{e}}$ and $\breve{\mathbf{X}}_k^{\mathrm{e}}$ can then be fed back to the filtering algorithm, i.e., they enter the subsequent prediction step.

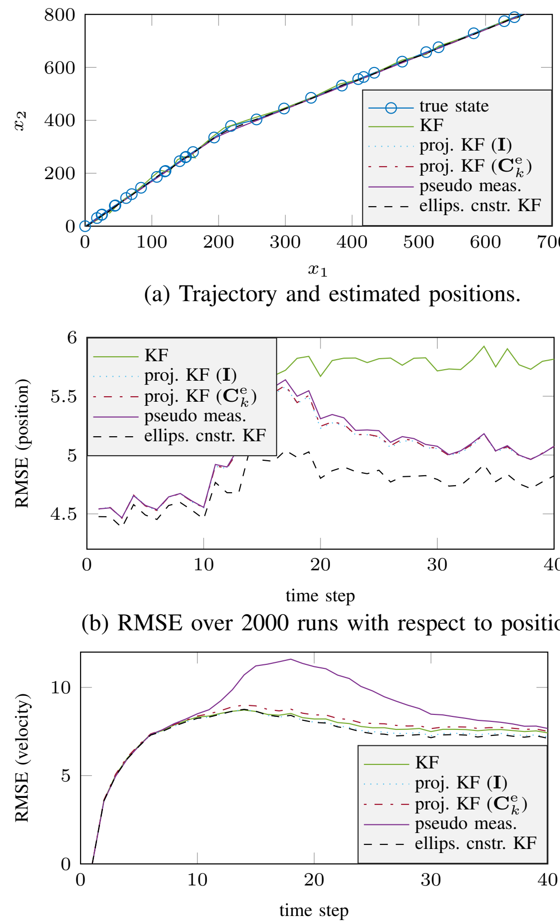
B. Discussion

Ellipsoidal state estimation: Ellipsoidal state estimation techniques [17], [19], [20] are closely related to the proposed concept. Such guaranteed estimation techniques utilize bounded sets to which the state, process and measurement uncertainties are confined. In this article, the state is governed by a stochastic model, and physical observations are affected by random, possibly unbounded errors; only the constraint is represented by an ellipsoidal set. This hybrid problem structure can be treated by the combined filter explained in Sec. III-B.

Batch processing: Both physical and pseudo measurements can be processed en-bloc. As in (18), the combined measurement equation

$$\begin{bmatrix} \hat{\underline{z}}_k \\ \breve{\underline{z}}_k \end{bmatrix} = \begin{bmatrix} \mathbf{H}_k \\ \breve{\mathbf{H}}_k \end{bmatrix} \underline{\mathbf{x}}_k + \begin{bmatrix} \underline{\mathbf{v}}_k \\ \underline{0} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \underline{\mathbb{d}}_k \end{bmatrix} \ , \tag{24}$$

can be considered. The measurement is affected by two noise terms, each of which is characterized by a singular matrix, i.e., $\left[\begin{smallmatrix} \mathbf{C}_k^{\mathrm{v}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_k^{\mathbb{d}} \end{smallmatrix}\right]$. By means of these parameters, a single update step according to (13) can be performed.



(a) Trajectory and estimated positions.

(b) RMSE over 2000 runs with respect to position.

(c) RMSE over 2000 runs with respect to velocity.

Fig. 1: Simulation results and comparison of different concepts.

Nonsingular matrices: As opposed to the concept of error-free pseudo measurements in (15), the proposed method does, in general, not result into singular covariance or shape matrices. As it can be seen from the combined model (24), the sum of both error matrices is nonsingular and hence, numerical problems can be circumvented.

Uncertainty: The processing of ellipsoidal constraints in (24) generally reduces the mean squared error. However, the size of the bounding ellipsoid, i.e., $\mathbf{X}_k^{\mathbb{d}}$, determines the effect on the mean squared error. As one might expect, large ellipsoidal constraints will lead to a reduced effect on the state estimate.

VII. Illustrative Example

In order to illustrate the proposed method, we revisit the example used in [10], which has also been studied in [8] and [7]. The state is four-dimensional with two-dimensional position and velocity components. The parameters for the system model (1) are

$$\mathbf{A}_k = \begin{bmatrix} 1 & 0 & Ts & 0 \\ 0 & 1 & 0 & Ts \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ , \quad \mathbf{B}_k = \begin{bmatrix} 0 \\ 0 \\ Ts \sin\theta_{k-1} \\ Ts \cos\theta_{k-1} \end{bmatrix} \ .$$

The control input $\hat{\underline{u}}_k$ is set to 1 as long as each velocity component is less than 30 m/s, and is altered to -1 until each velocity component is less than 5 m/s. Then, $\underline{\hat{u}}_k$ is again changed to 1. The process noise has the covariance matrix

$$\mathbf{C}_k^{\mathbf{w}} = \mathrm{diag}(400\,\mathrm{m}^2, 100\,\mathrm{m}^2, 16\,\mathrm{m}^2/\mathrm{s}^2, 1\,\mathrm{m}^2/\mathrm{s}^2) \ .$$

The initial state and estimate are

$$\begin{aligned}\underline{\hat{x}}_0^{\mathrm{e}} &= [0\mathrm{m}, 0\mathrm{m}, 10\tan\theta_0 \mathrm{m/s}, 10\mathrm{m/s}]^{\mathrm{T}} \ , \\ \mathbf{C}_0^{\mathrm{e}} &= \mathrm{diag}(400\,\mathrm{m}^2, 400\,\mathrm{m}^2, 10\,\mathrm{m}^2/\mathrm{s}^2, 10\,\mathrm{m}^2/\mathrm{s}^2) \ .\end{aligned}$$

The measurement model (2) is defined by

$$\mathbf{H}_k = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right] \ ,$$

and the measurement noise $\underline{\mathbf{v}}_k$ has the covariance matrix

$$\mathbf{C}_k^{\mathbf{v}} = \mathrm{diag}(18\mathrm{m}^2, 18\mathrm{m}^2) \ .$$

The vehicle is supposed to stay on straight road segments. As shown in Fig. 1(a), the road has an orientation of $60°$, which changes to $45°$ at the turning point. This constraint is represented by

$$\mathbf{D}_k = \begin{bmatrix} 1 & -\tan\theta_k & 0 & 0 \\ 0 & 0 & 1 & -\tan\theta_k \end{bmatrix} \ ,$$

with

$$\underline{d}_k = \begin{cases} [-1, 0]^{\mathrm{T}} & \text{for } \theta_k = 60° \ , \\ [147.4, 0]^{\mathrm{T}} & \text{for } \theta_k = 45° \ . \end{cases}$$

In order to model imperfect knowledge about the constraint, the vector $\underline{d}_k$ has a bias of $[1, 0]$. In comparison to [10], we have chosen different values for the parameters $\underline{d}_k$ and $\mathbf{C}_k^{\mathbf{v}}$ for the purpose of modeling the uncertainty about the constraint and a better measurement quality, respectively.

In Fig. 1, different estimators are compared after 2000 Monte-Carlo runs. The projection-method from Sec. IV-A uses the parameters $\mathbf{W} = \mathbf{I}$ and $\mathbf{W} = \mathbf{C}_k^{\mathrm{e}}$. Fig. 1(c) shows that an estimator based on pseudo measurements, as explained in Sec. IV-B, is highly susceptible to a constraint mismatch. With the aid of the estimator proposed in Sec. VI, the imperfect knowledge about the constraint can be represented by an ellipsoid $\mathcal{E}(0, \mathbf{X}_k^{\mathrm{d}})$ with $\mathbf{X}_k^{\mathrm{d}} = \mathrm{diag}(4, 0.16)$. The proposed estimation algorithm guarantees a lower root-mean-squared error (RMSE).

VIII. CONCLUSIONS

Equality constraints are often difficult to define precisely and are possibly too restrictive. In particular, an important issue is how to model and treat imprecise knowledge about the constraint. With ellipsoidal constraints, a bounded region around the constraint can be modeled, which can be chosen conservatively in order to take account of possibly incorrect constraint parameters. In many applications, a set-membership representation also appears to be most appropriate. For instance, the width of a road in ground vehicle-tracking applications can directly be interpreted as an interval, i.e., an one-dimensional ellipsoid. It has been demonstrated that ellipsoidal constraints can easily be treated within a generalized Kalman filter framework. Another advantage over equality constraints is that the error covariance and shape matrices, in general, do not become singular. Compared to inequality constraints, using ellipsoidal constraints only involves a simple optimization of a single parameter.

Prospective research will focus on combining ellipsoidal constraints with projection-based methods [8] and the concept of direct elimination [11]. The proposed concept will, in particular, be applied to nonlinear, ellipsoidal constraints.

REFERENCES

[1] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," *Transactions of the ASME - Journal of Basic Engineering*, vol. 82, pp. 35–45, 1960.

[2] C. Yang and E. Blasch, "Fusion of Tracks with Road Constraints," *Journal of Advances in Information Fusion*, vol. 3, no. 1, pp. 14–32, Jun. 2008.

[3] W. Wen and H. F. Durrant-Whyte, "Model-based Multi-sensor Data Fusion," in *Proceedings of the 1992 International Conference on Robotics and Automation (ICRA 1992)*, Nice, France, May 1992.

[4] D. Simon, *Optimal State Estimation: Kalman, H Infinity, and Nonlinear Approaches*. John Wiley & Sons, 2006.

[5] B. J. Spivey, J. D. Hedengren, and T. F. Edgar, "Constrained Nonlinear Estimation for Industrial Process Fouling," *Industrial & Engineering Chemistry Research*, vol. 49, no. 17, pp. 7824–7831, 2010.

[6] J. Zhou and Y. Zhu, "The Linear Minimum Mean-square Error Estimation with Constraints and Its Applications," in *Proceedings of the 2006 International Conference on Computational Intelligence and Security (CIS 2006)*, Guangzhou, China, Nov. 2006.

[7] S. Ko and R. R. Bitmead, "State Estimation for Linear Systems With State Equality Constraints," *Automatica*, vol. 43, no. 8, pp. 1363–1368, Aug. 2007.

[8] D. Simon and T. L. Chia, "Kalman Filtering with State Equality Contraints," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 38, no. 1, pp. 128–136, Jan. 2002.

[9] Z. Duan and X.-R. Li, "The Role of Pseudo Measurements in Equality-Constrained State Estimation," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 49, no. 3, pp. 1654–1666, Jul. 2013.

[10] C.-Y. Jiang and Y.-A. Zhang, "Some Results on Linear Equality Constrained State Filtering," *International Journal of Control*, vol. 86, no. 12, pp. 2115–2130, Jun. 2013.

[11] Z. Duan, X.-R. Li, and J. Ru, "Design and Analysis of Linear Equality Constrained Dynamic Systems," in *Proceedings of the 15th International Conference on Information Fusion (Fusion 2012)*, Singapore, Jul. 2012.

[12] D. Simon and D. L. Simon, "Kalman Filtering with Inequality Constraints for Turbofan Engine Health Estimation," *IEE Proceedings Control Theory and Applications*, vol. 153, no. 3, pp. 371–378, May 2006.

[13] D. Simon, "Kalman Filtering with State Constraints: A Survey of Linear and Nonlinear Algorithms," *IET Control Theory & Applications*, vol. 4, no. 8, pp. 1303–1318, Aug. 2010.

[14] B. O. Teixeira, L. A. Tôrres, L. A. Aguirre, and D. S. Bernstein, "On Unscented Kalman Filtering with State Interval Constraints," *Journal of Process Control*, vol. 20, no. 1, pp. 47–57, Jan. 2010.

[15] B. Noack, F. Pfaff, and U. D. Hanebeck, "Optimal Kalman Gains for Combined Stochastic and Set-Membership State Estimation," in *Proceedings of the 51st IEEE Conference on Decision and Control (CDC 2012)*, Maui, Hawaii, USA, Dec. 2012.

[16] B. Noack, *State Estimation for Distributed Systems with Stochastic and Set-membership Uncertainties*, ser. Karlsruhe Series on Intelligent Sensor-Actuator-Systems 14. Karlsruhe, Germany: KIT Scientific Publishing, 2013.

[17] A. B. Kurzhanski and I. Vályi, *Ellipsoidal Calculus for Estimation and Control*. Birkhäuser, 1997.

[18] R. J. Hewett, M. T. Heath, M. D. Butala, and F. Kamalabadi, "A Robust Null Space Method for Linear Equality Constrained State Estimation," *IEEE Transactions on Signal Processing*, vol. 58, no. 8, pp. 3961–3971, Aug. 2010.

[19] F. L. Chernousko, "Ellipsoidal State Estimation for Dynamical Systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 63, no. 5-7, pp. 872 – 879, Nov. 2005.

[20] F. L. Chernousko, *State Estimation for Dynamic Systems*. CRC Press, 1994.