Why You Shouldn't Use TDOA for Multilateration

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***Abstract*—The maximum likelihood problem arising from multilateration or source localization via signal times of arrival (TOA) leads to a nonlinear least squares problem in target position and target transmission time (TTT). Since we are not interested in the latter, it is usually eliminated from the equation system. We eliminate the TTT in closed form, which is simpler to design, easier to implement, and faster to compute than the often used pairwise time differences of arrival (TDOAs). We propose an unweighted nonlinear least squares formulation of the multilateration problem whose minimization with the Levenberg-Marquardt algorithm is very fast.**

***Index Terms*—source localization, multilateration, maximum likelihood, time of arrival (TOA), time difference of arrival (TDOA), Weighted nonlinear least squares, Levenberg-Marquardt**

Julia source code is available here (others added on request): https://github.com/KIT-ISAS/MFI2025_MLAT-TOA

I. Introduction

Multilateration is a method for passive source localization, i.e., a "listening-only" sensor passively receives signals (e.g., electromagnetic signals or sounds) that have propagated with known propagation speed (trough vacuum, air, or water) from the emitter. Based on time of arrival (TOA) measurements, the emitter location can be determined. This can be realized with a relatively cheap sensor infrastructure and can yield very high accuracy. A major application area is secondary surveillance radar (SSR), i.e., the tracking of cooperative aerial targets.

There are convenient algebraic methods providing a closed-form solution [1], [2], [3]. All these methods, however, introduce some nonlinear transformation of the measurement equation, usually to get rid of the square root from the Euclidean distance. This implies that the solution is no longer optimal under the presence of additive Gaussian measurement noise.

The "gold standard" maximum likelihood estimation requires an iterative search for target position and target transmission time (TTT) via numerical optimization. This is proposed in [4]. Some also require target and sensors being time-synchronized, i.e., the TTT to be known, so that one needs to search for target position only [5], [6].

The TTT can also be eliminated by taking differences of pairs of measurements or by solving a linear least squares problem, which we focus on in this work. The specific and somewhat subtle distinction from prior state of the art in these areas is given in Section III-E and Section IV-C, after having established the notation and problem structure.

II. Problem Description

A. Measurement Equation

A target at unknown location

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1}$$

transmits a radio signal at unknown time t_0. This signal is received by various sensors at known locations

$$\underline{s}_i = \begin{bmatrix} s_{i,x} \\ s_{i,y} \\ s_{i,z} \end{bmatrix}^\top , \quad \text{for } i \in [1, 2, \ldots, N] \ . \tag{2}$$

Each sensor measures the TOA t_i with zero-mean additive Gaussian noise v_i with stochastic properties

$$\mathrm{Var}\{v_i\} = \sigma_i^2 \ , \tag{3}$$

$$\mathrm{Cov}\{v_i, v_j\} = 0 \ \text{ for } i \neq j \ . \tag{4}$$

Therefore, we have the measurement equation

$$t_i = t_0 + \frac{1}{c} \|\underline{x} - \underline{s}_i\| + v_i \ , \tag{5}$$

where c is the signal propagation speed. In vector notation, for all measurements combined, it is

$$\underline{t} = t_0 \cdot \underline{1} + \underline{h}(\underline{x}) + \underline{v} \ , \tag{6}$$

where

$$\underline{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_N \end{bmatrix} , \quad \underline{h}(\underline{x}) = \frac{1}{c} \begin{bmatrix} \|\underline{x} - \underline{s}_1\| \\ \|\underline{x} - \underline{s}_2\| \\ \vdots \\ \|\underline{x} - \underline{s}_N\| \end{bmatrix} , \quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} , \tag{7}$$

with noise covariance

$$\mathrm{Cov}\{\underline{v}\} = \mathbf{C}_v = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_N^2 \end{bmatrix} \ . \tag{8}$$

B. Maximum Likelihood Estimation

The likelihood is therefore

$$f(\underline{t} \mid t_0, \underline{x}) = f_v(-t_0 \cdot \underline{1} + \underline{t} - \underline{h}(\underline{x})) \ , \tag{9}$$

with

$$f_v(\underline{v}) \propto \exp\left\{ -\frac{1}{2} \, \underline{v}^\top \mathbf{C}_v^{-1} \underline{v} \right\} \ . \tag{10}$$

Finally, the maximum likelihood (ML) solution is

$$(\hat{t}_0, \hat{\underline{x}}) = \underset{(t_0,\underline{x})}{\arg\max} f(\underline{t} \mid t_0, \underline{x}) \tag{11}$$

$$= \underset{(t_0,\underline{x})}{\arg\min} \|t_0 \cdot \underline{1} - \underline{t} + \underline{h}(\underline{x})\|^2_{\mathbf{C}_v^{-1}} \ , \tag{12}$$

where $\|\underline{x}\|^2_{\mathbf{C}} = \underline{x}^\top \mathbf{C} \underline{x}$. This can, for diagonal $\mathbf{C}_v$, also be written as

$$(\hat{t}_0, \hat{\underline{x}}) = \underset{(t_0,\underline{x})}{\arg\min} \sum_{i=1}^{N} \left[\frac{1}{\sigma_i} \left(t_0 - t_i + \frac{1}{c} \|\underline{x} - \underline{s}_i\| \right) \right]^2 \ . \tag{13}$$

This nonlinear least squares problem could readily be solved with the Gauss-Newton algorithm [7], [8], or its regularized variant, the the Levenberg-Marquardt algorithm [9], [10], or also Powell's hybrid "dog-leg" method [11], [12]. However, it seems desirable to first get rid of the unknown TTT t_0 that we are not interested in, to not burden the optimizer with unnecessary work.

III. TDOA Method

What is often done is to subtract two TOA measurement equations from each other, yielding the so-called time difference of arrival (TDOA) measurement equation

$$t_i - t_j = \frac{1}{c}\|\underline{x} - \underline{s}_i\| - \frac{1}{c}\left\|\underline{x} - \underline{s}_j\right\| + v_i - v_j \ , \tag{14}$$

which can also be written as

$$\tau_{i,j} = \eta_{i,j}(\underline{x}) + \nu_{i,j} \ , \tag{15}$$

$$\eta_{i,j}(\underline{x}) = \frac{1}{c}\|\underline{x} - \underline{s}_i\| - \frac{1}{c}\left\|\underline{x} - \underline{s}_j\right\| \ . \tag{16}$$

Two things have changed here: the TTT t_0 got eliminated, as desired, and the measurement noise changed from v_i to

$$\nu_{i,j} = v_i - v_j \ . \tag{17}$$

To again obtain the appropriate ML estimator, we have to compute the noise covariance of $\nu_{i,j}$

$$\mathrm{Cov}\{\nu_{i,j}, \nu_{k,l}\} = \mathrm{Cov}\{v_i - v_j, v_k - v_l\} \tag{18}$$

$$= \mathrm{E}\{v_i v_k\} - \mathrm{E}\{v_i v_l\} - \mathrm{E}\{v_j v_k\} + \mathrm{E}\{v_j v_l\} \ . \tag{19}$$

A. Consecutive Topology

Assuming we take differences of consecutive measurements, i.e., $N-1$ measurements $\tau_{i,j}$, in particular,

$$\underline{\tau} = \begin{bmatrix} \tau_{1,2} & \tau_{2,3} & \dots & \tau_{N-1,N} \end{bmatrix}^\top \ . \tag{20}$$

Then the covariance is

$$\mathbf{C}_\nu = \tag{21}$$

$$\begin{bmatrix} \sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & & & 0 \\ -\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_3^2 & & \\ & -\sigma_3^2 & \sigma_3^2 + \sigma_4^2 & \ddots & \\ & & \ddots & \ddots & -\sigma_{N-1}^2 \\ 0 & & & -\sigma_{N-1}^2 & \sigma_{N-1}^2 + \sigma_N^2 \end{bmatrix} , \tag{22}$$

because

$$\mathrm{Cov}\{\nu_{i,i+1}, \nu_{i,i+1}\} \tag{23}$$

$$= \mathrm{E}\{v_i v_i\} - \mathrm{E}\{v_i v_{i+1}\} - \mathrm{E}\{v_{i+1} v_i\} + \mathrm{E}\{v_{i+1} v_{i+1}\} \tag{24}$$

$$= \sigma_i^2 - 0 - 0 + \sigma_{i+1}^2 = \sigma_i^2 + \sigma_{i+1}^2 \ , \tag{25}$$

and

$$\mathrm{Cov}\{\nu_{i-1,i}, \nu_{i,i+1}\} \tag{26}$$

$$= \mathrm{E}\{v_{i-1} v_i\} - \mathrm{E}\{v_{i-1} v_{i+1}\} - \mathrm{E}\{v_i v_i\} + \mathrm{E}\{v_i v_{i+1}\} \tag{27}$$

$$= 0 - 0 - \sigma_i^2 + 0 = -\sigma_i^2 \ . \tag{28}$$

B. Star Topology

Alternatively, we could use a star-like topology, where we also get $N-1$ measurements $\tau_{i,j}$

$$\underline{\tau} = \begin{bmatrix} \tau_{1,2} & \tau_{1,3} & \dots & \tau_{1,N} \end{bmatrix}^\top \ . \tag{29}$$

In that case the covariance is

$$\mathbf{C}_\nu = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & & & \sigma_1^2 \\ & \sigma_1^2 + \sigma_3^2 & & \\ & & \ddots & \\ \sigma_1^2 & & & \sigma_1^2 + \sigma_N^2 \end{bmatrix} , \tag{30}$$

because

$$\mathrm{Cov}\{\nu_{1,i}, \nu_{1,j}\} = \sigma_1^2 + \mathrm{E}\{v_i v_j\} \tag{31}$$

$$= \begin{cases} \sigma_1^2 + \sigma_i^2 \ , & i = j \\ \sigma_1^2 \ , & i \neq j \end{cases} \ . \tag{32}$$

C. Combinatorial

One may also decide to take all combinations of two measurements. Then $\underline{\tau}$ has

$$\binom{N}{2} = \frac{N \cdot (N-1)}{2} \tag{33}$$

elements, and $\mathbf{C}_\nu$ must be individually compiled from (18) for the respective N and ordering of the combinations.

D. Solver

To obtain the ML solution, we have to solve a weighted nonlinear least squares problem similar to (12) but adapted to the transformed measurements $\tau_{i,j}$

$$\hat{\underline{x}} = \underset{\underline{x}}{\arg\min} \left\|\underline{\eta}(\underline{x}) - \underline{\tau}\right\|^2_{\mathbf{C}_\nu^{-1}} \ , \tag{34}$$

with $\underline{\eta}(\underline{x})$ the vectorized version of $\eta_{i,j}(\underline{x})$ (16). Now we have a non-diagonally weighted nonlinear least squares problem. Although the Levenberg-Marquardt algorithm can be derived for such problems [13], [14, p. 515], the major implementations do not give the option to provide a weighting matrix. We can, however, rewrite the quadratic form in (34) into a sum of squares [15, p. 6] via the Cholesky decomposition $\mathbf{C}_\nu = \mathbf{R}\mathbf{R}^\top$

$$\left\|\underline{\eta}(\underline{x}) - \underline{\tau}\right\|^2_{\mathbf{C}_\nu^{-1}} \tag{35}$$

$$= \left[\underline{\eta}(\underline{x}) - \underline{\tau}\right]^\top \left(\mathbf{R}\mathbf{R}^\top\right)^{-1} \left[\underline{\eta}(\underline{x}) - \underline{\tau}\right] \tag{36}$$

$$= \left[\underline{\eta}(\underline{x}) - \underline{\tau}\right]^\top \mathbf{R}^{-\top} \mathbf{R}^{-1} \left[\underline{\eta}(\underline{x}) - \underline{\tau}\right] \tag{37}$$

$$= \left[\mathbf{R}^{-1}\left(\underline{\eta}(\underline{x}) - \underline{\tau}\right)\right]^\top \left[\mathbf{R}^{-1}\left(\underline{\eta}(\underline{x}) - \underline{\tau}\right)\right] \ . \tag{38}$$

Thus, transforming the vector $(\underline{\eta}(\underline{x}) - \underline{\tau})$ with $\mathbf{R}^{-1}$ renders the least squares problem unweighted. However, this matrix multiplication makes computation of the objective function noticeably slower and can be avoided by our proposed method.

E. Distorted Solvers

It seems that some TDOA users do not take into account the correlations between the $\nu_{i,j}$ and simply treat the TDOA measurements as if they had uncorrelated noise, i.e., with $\mathbf{C}_\nu$ diagonal [16, Eq. 29], [17, Eq. 10-11], [18, Eq. 7], [19, Eq. 10], [20, Eq. 5-6]. This may be justified in some cases, namely when time differences are directly measured, e.g., via cross-correlation of the two received waveforms [21], [22], [23]. But usually, the TOA are determined separately in each sensor, and only subsequently the resulting timestamps being subtracted, yielding the "TDOA" values, which should actually better be called difference of times of arrival (DOTA) [15]. Some users, however, do correctly respect the TDOA correlations [24, Eq. 44], [25, Eq. 46], [26, Eq. 33], [14, p. 514], [27, Eq. 3], [15, Fig. 2], [28, Eq. 8]. Finally, some aim to respect correlations but use the wrong covariance matrix [29, Eq. 11], [30].

IV. Proposed TOA Method

We claim that using differences of TOAs unnecessarily complicates things (in particular, the covariance matrix), and instead propose a different way dealing with the "unwanted unknown" t_0, which does not lead to a non-diagonal weighting matrix.

A. Key Idea

Note that the measurement equation (5) is affine in t_0. Thus, if $\underline{x}$ was given, we could solve for t_0 via linear least squares, i.e., in closed form. And that is exactly what we propose: let the nonlinear solver, just like for TDOA, propose some value for $\underline{x}$ and not for t_0. Then for that specific $\underline{x}$, compute the optimal t_0 and return the resulting loss as objective function.

B. Computing t_0

In particular, we simply solve the vectorized measurement equation (6) for t_0. First, we rearrange it from being affine in t_0 to being linear in t_0

$$\underline{t} - \underline{h}(\underline{x}) = \underline{1} \cdot t_0 + \underline{v} \ . \tag{39}$$

For normally distributed $\underline{v}$ we obtain the ML estimate $\hat{t}_0$ via linear least squares

$$\hat{t}_0(\underline{x}) = \left(\underline{1}^\top \mathbf{C}_v^{-1} \underline{1}\right)^{-1} \cdot \underline{1}^\top \mathbf{C}_v^{-1} (\underline{t} - \underline{h}(\underline{x})) \ , \tag{40}$$

which for diagonal $\mathbf{C}_v$ as in (8), becomes

$$\hat{t}_0(\underline{x}) = \frac{\sum_{i=1}^N \frac{1}{\sigma_i^2} \left[t_i - \frac{1}{c}\|\underline{x} - \underline{s}_i\|\right]}{\sum_{i=1}^N \frac{1}{\sigma_i^2}} \ , \tag{41}$$

and for $\mathbf{C}_v = \sigma^2 \mathbf{I}$

$$\hat{t}_0(\underline{x}) = \frac{1}{N} \sum_{i=1}^N \left[t_i - \frac{1}{c}\|\underline{x} - \underline{s}_i\|\right] \ . \tag{42}$$

Thus, $\hat{t}_0$ is simply the weighted sample mean of

$$t_i - \frac{1}{c}\|\underline{x} - \underline{s}_i\| \ . \tag{43}$$

Inserting this into (6) yields a measurement equation that depends only on $\underline{x}$.

C. State-of-Art

A similar approach has already been proposed in [31, Eq. 8+10]. However, they include a penalty that keeps the t_0-estimate of the next iteration close to the current one, which we deem unnecessary. Furthermore it requires an initial guess of t_0 which our method does not. But most of all, they treat finding $\hat{t}_0$ and $\hat{x}_0$ as two entirely separate optimizations problems that are executed alternately. Thus, after each linear least squares $\underline{t}_0$ estimation, they perform a *multi-iteration* nonlinear optimization solving for $\underline{x}$ with the same $\underline{t}_0$ [31, Alg. 1]. Lastly, they propose a gradient-descent optimization, while we propose, exploiting the problem structure, the Levenberg-Marquardt method for the variant described in Section IV-D, or quasi-Newton for Section IV-E.

Similarly, [32, Eq. 4], [33] use the closed-form $\underline{t}_0$-estimate (41) but solve the entire problem via semidefinite programming, which is generally slower. Also [34, Eq. 6] uses this trick – but still arrives at a non-diagonally weighted least squares problem [34, Eq. 7+A7], just like we do for the correctly-weighted TDOA problem (36) (with the associated somewhat higher computational cost) and unlike our diagonally-weighted least squares problem for TOA (45).

D. Levenberg-Marquardt

Inserting (41) into (13) gives a sum-of-squares minimization problem that depends only on $\underline{x}$ and can be solved with the Levenberg-Marquardt algorithm.

Objective Function: Levenberg-Marquardt requires an objective function θ_{LM} that returns a vector of the terms to be squared and summed, which would be

$$\underline{\theta}_{\mathrm{LM}}(\underline{x}) = \begin{bmatrix} \frac{1}{c}\|\underline{x} - \underline{s}_1\| + \hat{t}_0(\underline{x}) - t_1 \\ \frac{1}{c}\|\underline{x} - \underline{s}_2\| + \hat{t}_0(\underline{x}) - t_2 \\ \vdots \\ \frac{1}{c}\|\underline{x} - \underline{s}_N\| + \hat{t}_0(\underline{x}) - t_N \end{bmatrix} \ , \tag{44}$$

with $\hat{t}_0(\underline{x})$ from (41), where

$$\hat{\underline{x}} = \arg\min_{\underline{x}} [\underline{\theta}_{\mathrm{LM}}(\underline{x})]^\top \mathbf{C}_v^{-1} [\underline{\theta}_{\mathrm{LM}}(\underline{x})] \ . \tag{45}$$

Jacobian: Furthermore, Levenberg-Marquardt needs the Jacobian of $\underline{\theta}_{\mathrm{LM}}(\underline{x})$. First, we define the Jacobian of $\underline{h}(x)$, $\mathbf{J}_{\underline{h}}(\underline{x}) \in \mathbb{R}^{N\times 3}$

$$\mathbf{J}_{\underline{h}} = \frac{1}{c} \begin{bmatrix} (\underline{x} - \underline{s}_1)^\top / \|x - \underline{s}_1\| \\ (\underline{x} - \underline{s}_2)^\top / \|x - \underline{s}_2\| \\ \vdots \\ (\underline{x} - \underline{s}_N)^\top / \|x - \underline{s}_N\| \end{bmatrix} \tag{46}$$

Method	Weighted	Unweighted
TDOA-Consecutive	229 µs	190 µs
TDOA-Star	228 µs	198 µs
TOA (proposed)	unnecessary	156 µs

TABLE I: Computation times for the setup described in Section VI-A.

Method	Weighted	Unweighted
TDOA-Consecutive	0.370	0.452
TDOA-Star	0.370	0.804
TOA (proposed)	–	0.370

TABLE II: Median Euclidean deviation of the results for the setup described in Section VI-C.

and weight vector $\underline{w}$ containing the diagonal elements of the diagonal $\mathbf{C}_v^{-1}$

$$\underline{w} = \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{1}{\sigma_2^2} & \cdots & \frac{1}{\sigma_N^2} \end{bmatrix}^\top \ . \tag{47}$$

Then the derivative of $\hat{t}_0(\underline{x})$ from (41), $\frac{\partial \hat{t}_0(\underline{x})}{\partial \underline{x}^\top} \in \mathbb{R}^{1\times 3}$, with included σ_i, is

$$\frac{\partial \hat{t}_0(\underline{x})}{\partial \underline{x}^\top} = -\frac{1}{\underline{1}^\top \underline{w}} \cdot \underline{w}^\top \mathbf{J}_{\underline{h}}(\underline{x}) \ . \tag{48}$$

The desired Jacobian of θ_{LM} from (44), $\mathbf{J}_{\underline{\theta}_{\mathrm{LM}}}(\underline{x}) \in \mathbb{R}^{N\times 3}$, is

$$\mathbf{J}_{\underline{\theta}_{\mathrm{LM}}}(\underline{x}) = \sqrt{\mathrm{diag}(\underline{w})} \cdot \left(\mathbf{J}_{\underline{h}}(\underline{x}) - \frac{1}{\underline{1}^\top \underline{w}} \cdot \underline{w}^\top \mathbf{J}_{\underline{h}}(\underline{x}) \right) \ . \tag{49}$$

Alternatively, the Jacobian may be computed via Automatic Differentiation if supported by the respective language.

E. As Sample Variance Computation

Consider again the nonlinear least squares objective function, slightly modified from (13)

$$\theta(\underline{x}) = \frac{1}{\sum_{i=1}^N \frac{1}{\sigma_i^2}} \sum_{i=1}^N \frac{1}{\sigma_i^2} \left[\frac{1}{c} \|\underline{x} - \underline{s}_i\| - t_i - (-\hat{t}_0(\underline{x})) \right]^2 \tag{50}$$

and compare it to our linear least squares estimate $\hat{t}_0$ from (41)

$$-\hat{t}_0(\underline{x}) = \frac{1}{\sum_{i=1}^N \frac{1}{\sigma_i^2}} \sum_{i=1}^N \frac{1}{\sigma_i^2} \left[\frac{1}{c} \|\underline{x} - \underline{s}_i\| - t_i \right] \ . \tag{51}$$

Inserting (51) into (50) shows that what we have here is precisely a sample variance computation. This has two implications. First, we can also write it as

$$\theta(\underline{x}) = \frac{1}{\sum_{i=1}^N \frac{1}{\sigma_i^2}} \sum_{i=1}^N \frac{1}{\sigma_i^2} \left[\frac{1}{c} \|\underline{x} - \underline{s}_i\| - t_i \right]^2 \tag{52}$$

$$- \left(\frac{1}{\sum_{i=1}^N \frac{1}{\sigma_i^2}} \sum_{i=1}^N \frac{1}{\sigma_i^2} \left[\frac{1}{c} \|\underline{x} - \underline{s}_i\| - t_i \right] \right)^2 \ . \tag{53}$$

Second, we can simply use a readily available, highly optimized, and fast sample variance computation function to compute the weighted variance of the vector

$$\underline{h}(\underline{x}) - \underline{t} = \begin{bmatrix} \frac{1}{c}\|\underline{x} - \underline{s}_1\| - t_1 \\ \frac{1}{c}\|\underline{x} - \underline{s}_2\| - t_2 \\ \vdots \\ \frac{1}{c}\|\underline{x} - \underline{s}_N\| - t_N \end{bmatrix} \ . \tag{54}$$

Say, $\mathrm{SVar}(\underline{y};\ \underline{w})$ computes the weighted sample variance of a vector of univariate samples $\underline{y}$, then our desired objective function can be written as

$$\theta(\underline{x}) = \mathrm{SVar}(\underline{h}(\underline{x}) - \underline{t};\ \underline{w}) \ , \tag{55}$$

Consequently, we can interpret the nonlinear least squares solution

$$\hat{\underline{x}} = \arg\min_{\underline{x}} \theta(\underline{x}) \tag{56}$$

as the $\underline{x}$ that minimizes the sample variance of $(\underline{h}(\underline{x}) - \underline{t})$.

Gradient: The gradient of the objective function is

$$\frac{\partial \theta(\underline{x})}{\partial \underline{x}^\top} = \frac{2}{\underline{1}^\top \underline{w}} \cdot \underline{w}^\top \, \mathrm{diag}(\underline{h}(\underline{x}) - \underline{t}) \, \mathbf{J}_{\underline{h}}(\underline{x}) \tag{57}$$

$$- \frac{2}{(\underline{1}^\top \underline{w})^2} \cdot \left(\underline{w}^\top (\underline{h}(\underline{x}) - \underline{t}) \right) \cdot \left(\underline{w}^\top \mathbf{J}_{\underline{h}}(\underline{x}) \right) \ . \tag{58}$$

This can be seen as twice the weighted sample covariance between $(\underline{h}(\underline{x}) - \underline{t})$ and $\mathbf{J}_{\underline{h}}$. But computation with available covariance computation functions would have to be done separately for the three columns of $\mathbf{J}_{\underline{h}}$ such that both samples have the same dimension, one. Either way, with this we can now also solve the three-dimensional TOA multilateration problem with, e.g., a Quasi-Newton algorithm.

V. TL;DR – Minimal Implementation

Implement the TTT estimator $\hat{t}_0(\underline{x})\colon \mathbb{R}^3 \mapsto \mathbb{R}$ (42), using the measured TOAs t_i and sensor locations $\underline{s}_i$. Implement the objective function $\theta_{\mathrm{LM}}(\underline{x})\colon \mathbb{R}^3 \mapsto \mathbb{R}^N$ (44). Minimize $\theta_{\mathrm{LM}}(\underline{x})$ with the Levenberg-Marquardt algorithm, using forward differences for the gradients. This is also what you can find in our GitHub repository.

VI. Evaluation

A. Setup

For evaluation, we place $N = 100$ sensors $\underline{s}_i$ randomly in $[0, 10]^3$, define a ground truth, $t_0 = 0.2$ and $\underline{x} = [3, 1, 5]^\top$, assume $c = 1$, compute the noise-free measurements and add standard normal noise ($\sigma = 1$). We define the initial guess $\underline{x}_0 = [9, 8, 2]^\top$ and optimize in Julia using LeastSquaresOptim.jl, with standard settings (x_tol = f_tol = g_tol = 10^{-8}), using in-place syntax for the objective function and automatic differentiation via ForwardDiff.jl [35] for the Jacobian.

B. Timing

Computation times are shown in Table I. Thus, with the proposed TOA we have a computational load comparable than than unweighted TDOA – or even somewhat lower.

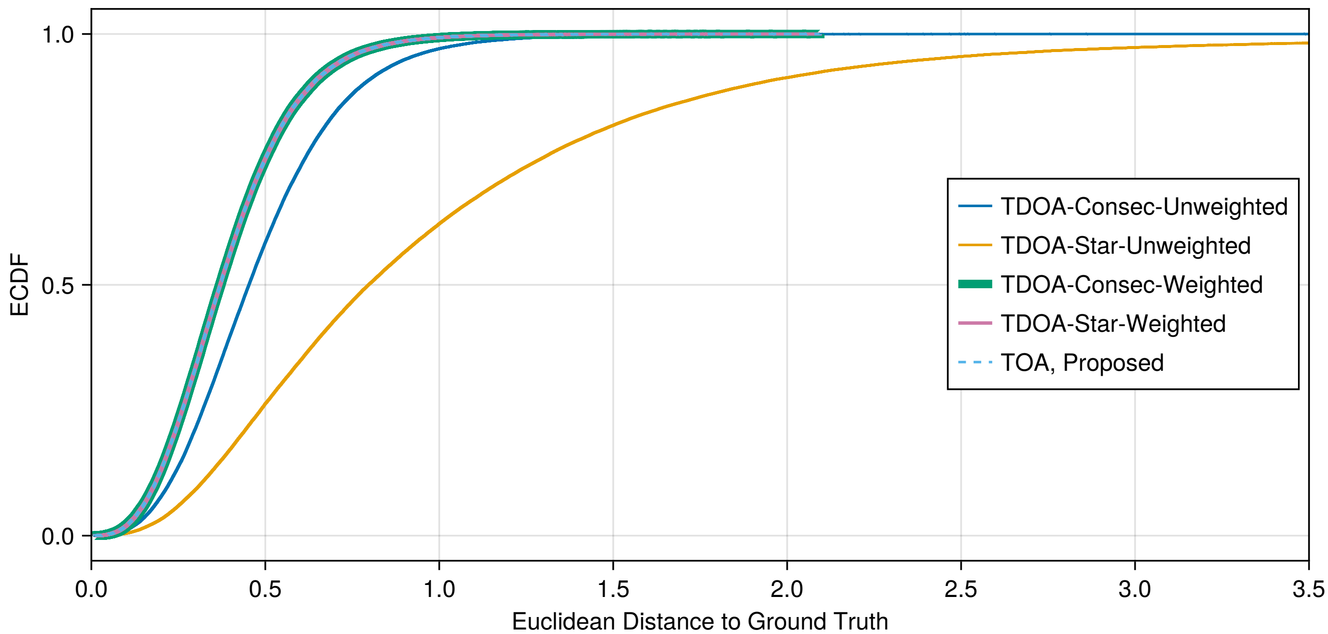

Fig. 1: Evaluation of our proposed TOA multilateration algorithm against four state-of-art methods. The correctly weighted TDOA methods as well as the proposed TOA method identically give the optimal results.

C. Accuracy

Now we randomly vary the sensor locations, repeating the experiment 100,000 times. The root mean square error (RMSE) values are listed in Table II. See Fig. 1 for a visualization of the distribution of the errors. All TDOA methods with the correct weighting matrix as well as the TOA method produce identical results.

D. Interpretation

The proposed TOA method has a runtime comparable than the unweighted TDOA methods – while producing the same result as the correctly weighted TDOA.

VII. Conclusion

We proposed a novel combination of unweighted least-squares objective function and Levenberg-Marquardt solver for the exact maximum likelihood problem in multilateration, yielding a very fast, efficient estimator. State-of-the-art methods use differences of pairs of TOAs, called TDOAs, but then the least squares problem turns into a weighted one – with non-diagonal covariance matrix, which is easily forgotten in design, error-prone in implementation, and slow in runtime. Our method is purely TOA-based, preserves the diagonal weighting matrix, and is faster to compute.

In particular, we derive a vector-valued objective function that is very simple to implement in-place and is suitable for the Levenberg-Marquardt algorithm (45), as well as a scalar-valued one that can be computed via a sample variance routine and then be optimized with a quasi-Newton method (56).

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