Localization of a Mobile Robot using Relative Bearing Measurements

K. Briechle and U. D. Hanebeck

Abstract— **In this article, the problem of robot localization based on relative bearing measurements is considered, where unknown but bounded measurement uncertainties are assumed. The standard approach is to linearize the nonlinear measurement equations and assume a simple shaped bounding set for the exact resulting set of states. In the new approach presented here, a nonlinear transformation of the measurement equation into a higher dimensional space is performed. This yields a tight closed–form nonlinear representation of the bounding set which is superior to commonly used bounding ellipsoids or box–shaped approximations.**

Keywords— **Robot Localization, Angle Measurements, Set–theoretic Estimation, Bounded Uncertainty and Errors in Variables.**

I. Introduction

Localization with respect to known features in the environment is one of the most important skills required for mobile robot navigation [1], [2], [3]. Based on measurements to these features, the so–called landmarks, the position and orientation of a mobile robot is determined with respect to a reference frame. We consider the relative bearing problem, where angular measurements to landmarks are available in simulations [4] or from omnidirectional vision sensors [5], [6]. Measurement errors due to sensor noise, landmark misidentification or inaccurate world models are usually modelled in a statistical framework [7], [8], [9]. Standard estimation tools like linear least squares [10], Extended Kalman filtering [11], [12] or more robust filters based on covariance intersection [13], [14] or particle filtering [9] can then be applied. However, the underlying statistical assumptions are often hard to verify [15] and parameters of the noise models have to be tuned.

To achieve a localization result which *guarantees* to contain all feasible values of the estimated robot pose consistent with the given measurements and prior knowledge, similar to [4], a bounded–error model is adopted. Within this framework, the only assumption on the measurement errors is, that they are bounded in amplitude [16], [17]. We assume, that the matching between landmarks in the map and measurements has been successfully performed, e. g., by tree search methods described in [18]. Furthermore, it is reasonable to assume, that errors due to map inaccuracies are small compared to the errors in the bearing measurements [4]. The desired result of an optimal localization algorithm for this error model is the set $\boldsymbol{\mathcal{X}}_k$ of all feasible robot positions compatible with all measurements. $\boldsymbol{\mathcal{X}}_k$ is guaranteed to contain the true but unknown robot position $\underline{x}_R$.

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As the complexity of a straightforward geometric evaluation of $\boldsymbol{\mathcal{X}}_k$ is dependent on the number of measurements, a conservative approximation $\boldsymbol{\mathcal{X}}_k^e$ of the exact set $\boldsymbol{\mathcal{X}}_k$ is required to solve the localization problem. This approximation should be as tight as possible, suitable for recursive filtering, and should be described by of a constant number of parameters.

In [4], two approximation techniques have been proposed based on previous research [19] in the set membership estimation area. These techniques are based on boxes and parallelotopes and yield a conservative approximation $\boldsymbol{\mathcal{X}}_k^e$ of the exact set of feasible positions $\boldsymbol{\mathcal{X}}_k$, which contains the true position of the robot. This approximation is calculated based on the relative bearing measurements and their associated error bounds. These simple–shaped sets can be computed with very little computational cost [4]. However, the drawback of this approach is, that the complex shaped exact sets $\boldsymbol{\mathcal{X}}_k$, which bound the position of the robot, can only roughly be approximated. Recent approaches proposed in [20], [21] calculate tighter bounds by set–inversion algorithms using interval analysis. The desired set of all feasible robot positions is approximated by inner and outer subpavings, each consisting of a list of nonoverlapping boxes.

This paper presents a simple, closed–form solution for the stated localization problem in the case of bounded errors, applying a new nonlinear filter concept [22], [23], that has been successfully applied to GSM mobile phone localization [24], and modified for the application presented here. It extends a concept based on overparametrization presented in [15] and has been applied to various nonlinear measurement types like distance measurements and angular measurements, though only relative bearing measurements are presented here as a special case. In this paper, we only consider the measurement step which is equivalent to localization of a static observer. A solution for the prediction step required for dynamic setups also exists.

The proposed nonlinear filtering concept allows to determine a conservative approximation of the desired set $\boldsymbol{\mathcal{X}}_k$, which is close to optimal even for non–convex sets $\boldsymbol{\mathcal{X}}_k$. The key idea of the proposed new approach is to expand the nonlinear measurement equation resulting from the relative bearing measurements to a pseudo–linear form, i. e., a form that is linear in a higher dimensional space S^* of transformed state variables. To achieve a tight approximation of the exact set of states $\boldsymbol{\mathcal{X}}_k$, a nonlinear transformation $\underline{\eta}_k(.)$ is applied to the measurement equation. This results in *nonlinear constraints* for the exact set of states $\boldsymbol{\mathcal{X}}_k$ in the original space S.

The paper is structured as follows. In Section II, a brief formulation of the problem of robot localization based on angular measurements is given. Section III-A reviews the standard bounding ellipsoidal filter and Section III-B introduces the concept of nonlinear filtering based on pseudo–ellipsoidal sets. Section IV describes the exact transformation of the given localization problem into a form, that is amenable to the proposed framework of nonlinear filtering. The new algorithm is applied to the localization problem in Section V.

II. Localization based on relative bearing measurements

We consider the localization problem for a vehicle navigating a 2–D environment. The vehicle pose at discrete time instant t_k is given by

$$\underline{x}_k = [x_k, y_k]^T \quad , \tag{1}$$

where x_k, y_k are the position coordinates of the vehicle with respect to a given reference frame. The vehicle is equipped with sensors, that provide relative angular measurements $\gamma_{i,j}(k)$ to pairs of landmarks L_i, L_j, $i,j = 1 \ldots N$ in the environment. Each measurement is subject to additive noise $v_k^{i,j}$ and related to the position $\underline{x}_k$ according to

$$\begin{aligned}\hat{\gamma}_{i,j}(k) =& \text{atan2}(y_k - y_{Li}, x_k - x_{Li}) - \\ & \text{atan2}(y_k - y_{Lj}, x_k - x_{Lj}) + v_k^{i,j} \quad .\end{aligned} \tag{2}$$

The measurement noise $v_k^{i,j}$ is assumed to be unknown but bounded according to

$$|v_k^{i,j}| \leq \epsilon_{i,j,k}^v, \quad i = 1 \ldots N \quad . \tag{3}$$

The coordinates $\underline{x}_{Li} = [x_{Li}, y_{Li}]^T$ of the N landmarks are assumed to be known. The localization problem as stated above is to determine the set $\boldsymbol{\mathcal{X}}_k$ of all robot positions $\underline{x}_k$ compatible with the relative angular measurements according to (2).

Each relative angular measurement $\gamma_{i,j}(k)$ to two landmarks L_i, L_j defines a circular arc $\boldsymbol{\mathcal{C}}_k^{i,j}$, which constrains the position of the vehicle, provided that the measurements are exact. The endpoints of the arc are the two given landmarks. Given an uncertain relative angular measurement $\hat{\gamma}_{i,j}(k) = \gamma_{i,j}(k) + v_k^{i,j}$, the set of feasible vehicle positions $\boldsymbol{\mathcal{M}}_k^{i,j}$ under assumption (3) is given by

$$\boldsymbol{\mathcal{M}}^{i,j} = \{\underline{x} \in \mathrm{I\!R}^2 : \hat{\gamma}_{i,j} - \epsilon_{i,j}^v \leq \gamma_{i,j} \leq \hat{\gamma}_{i,j} + \epsilon_{i,j}^v\} \quad ,$$

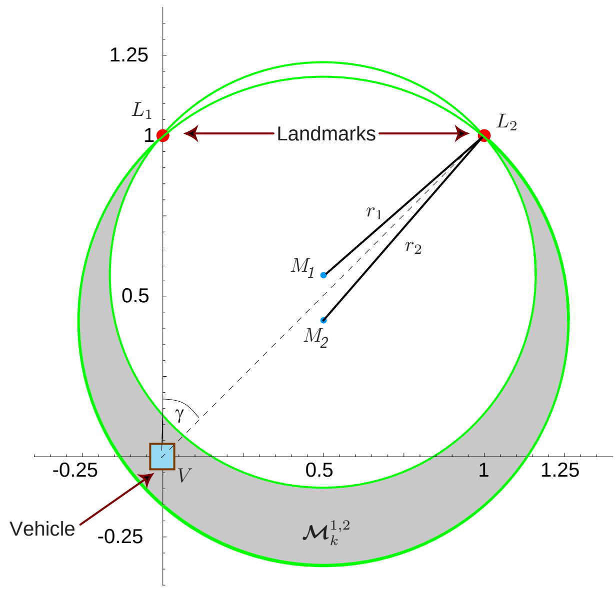

Fig. 1. Localization by two angle measurements with bounded uncertainties. $\boldsymbol{\mathcal{M}}_k^{1,2}$ is the exact set solely defined by the two angle measurements. The relative bearing angle is $\gamma = 45^o$ and the measurement uncertainty was chosen as $\epsilon_{1,2,k}^v = 4^o$.

where the time index k was omitted for clarity of notation. As outlined in [4], $\boldsymbol{\mathcal{M}}_k^{i,j}$ can be described as a "thickened ring" from a geometric point of view. In Fig. 1, an example for a resulting measurement set $\boldsymbol{\mathcal{M}}_k^{1,2}$ is shown for two landmarks L_1 and L_2 and a vehicle V at position $\underline{x}_k = [0,0]^T$. The measured difference angle in this example is $\gamma_{1,2} = 45^o$ and the upper bound for the measurement noise was chosen as $\epsilon_{1,2,k}^v = 4^o$.

Given a set of relative angular measurements $\gamma_{i,j}(k)$, $i,j = 1 \ldots N$, it is obvious, that the measurement set of feasible vehicle poses $\boldsymbol{\mathcal{X}}_k^M$ is defined by the intersection of all sets $\boldsymbol{\mathcal{M}}_k^{i,j}$

$$\boldsymbol{\mathcal{X}}_k^M = \bigcap_{i,j=1}^{N} \boldsymbol{\mathcal{M}}_k^{i,j} \quad .$$

Because $\boldsymbol{\mathcal{X}}_k^M$ is in general a complex shaped set, a conservative, approximative description $\tilde{\boldsymbol{\mathcal{X}}}_k^M$ of $\boldsymbol{\mathcal{X}}_k^M$ with $\tilde{\boldsymbol{\mathcal{X}}}_k^M \supset \boldsymbol{\mathcal{X}}_k^M$ is required, which can be used in a recursive estimation scheme. This approximation should be described by a finite set of parameters and should degrade gracefully with a decreasing number of parameters. In Fig. 8, an example for a resulting measurement set $\boldsymbol{\mathcal{X}}_k^M$ is shown for the case of $N = 3$ relative bearing measurements to three landmarks L_1, L_2 and L_3. It can be seen, that the gray–shaded set $\boldsymbol{\mathcal{X}}_k^M$ results from the intersection of three sets $\boldsymbol{\mathcal{M}}_k^{i,j}$. Note that this non–convex set cannot be represented by polytopes. To find an approximation $\tilde{\boldsymbol{\mathcal{X}}}_k^M$ of $\boldsymbol{\mathcal{X}}_k^M$ is in general not a trivial task.

In the proposed approach for localization in the case of relative bearing measurements, the problem stated in this Section is reformulated as a filtering problem in a system theoretic framework. Application of a recursive nonlinear filtering algorithm allows to sequentially fuse measurements with constant computational complexity, yielding an estimated set $\boldsymbol{\mathcal{X}}_k$ that is guaranteed to contain the true state $\underline{x}_k$ under the given assumptions. The algorithm yields estimates with small remaining approximation errors even for complex–shaped, non–convex sets, only depending on the approximation order.

The proposed new approach consists of two key ideas:The *first key idea* is to apply a new concept for nonlinear filtering, where a complicated uncertainty set $\boldsymbol{\mathcal{X}}_k$ in the N–dimensional original space S is represented by a simpler shaped uncertainty set $\boldsymbol{\mathcal{X}}_k^*$ in an L–dimensional hyperspace S^* with $L > N$. Similar to [15], the concept is based on overparametrization, but generalizes this idea in a system theoretic framework, which yields a simple, intuitive description of the nonlinear filtering problem.

The *second key idea* is to transform the given localization problem *exactly* into a form amenable to application of a general framework for nonlinear filtering, which is derived in Section IV. Equivalent transformations have been found for other types of nonlinear measurement equations, like bearing measurements or distance measurements [24].

III. Framework for Nonlinear Filtering

To apply the proposed framework for nonlinear filtering in the case of bounded error models, the measurement update equations of the standard bounding ellipsoid filter are used in a hyperspace S^* of transformed state variables to calculate a set $\boldsymbol{\mathcal{X}}_k^{e,*}$ in S^* that defines the complex–shaped estimated set of states $\boldsymbol{\mathcal{X}}_k^e$ in the original space S.

The equations of the standard bounding ellipsoid filter resemble the well–known Kalman filter equations. They were first derived in [17] with extensions from [25] for minimum volume ellipsoids and will be briefly reviewed in Section III-A. Section III-B derives the concept for nonlinear filtering using the standard bounding ellipsoid filter as a basic building block. This basic building block may be exchanged by, e. g., stochastic filtering algorithms [26] when uncertainties are modelled in a stochastic framework.

A. Standard Bounding Ellipsoidal Filter

At time step k, a prior estimate of the state $\underline{x}_k \in \mathbb{R}^{\mathrm{N}}$ described by the ellipsoidal set

$$\boldsymbol{\mathcal{X}}_k^p = \{\underline{x}_k : (\underline{x}_k - \underline{\hat{x}}_k^p)^T (\boldsymbol{C}_k^p)^{-1} (\underline{x}_k - \underline{\hat{x}}_k^p) \leq 1\} \ , \quad (4)$$

where $\boldsymbol{C}_k^p$ is a positive, symmetric matrix and $\underline{\hat{x}}_k^p$ is the midpoint vector, and a linear time varying measurement equation with uncertain measurement $\underline{\hat{z}}_k \in \mathbb{R}^{\mathrm{M}}$ according to

$$\underline{\hat{z}}_k = \boldsymbol{H}_k \underline{x}_k + \underline{v}_k \ , \quad (5)$$

with $M \times N$–dimensional measurement matrix $\boldsymbol{H}_k$ and additive, bounded measurement noise $\underline{v}_k$ are given. Then an ellipsoidal conservative approximation for the set of all states compatible with the measurement and the prior estimated set is obtained as

$$\boldsymbol{\mathcal{X}}_k^e = \{\underline{x}_k : (\underline{x}_k - \underline{\hat{x}}_k^e)^T (\boldsymbol{C}_k^e)^{-1} (\underline{x}_k - \underline{\hat{x}}_k^e) \leq 1\} \ , \quad (6)$$

where midpoint vector $\underline{\hat{x}}_k^e$ of the bounding ellipsoid $\boldsymbol{\mathcal{X}}_k^e$ is given by

$$\begin{aligned} \underline{\hat{x}}_k^e &= \underline{\hat{x}}_k^p + \lambda_k \boldsymbol{C}_k^p (\boldsymbol{H}_k)^T \\ &\quad \left\{\boldsymbol{V}_k + \lambda_k \boldsymbol{H}_k \boldsymbol{C}_k^p (\boldsymbol{H}_k)^T\right\}^{-1} (\underline{\hat{z}}_k - \boldsymbol{H}_k \underline{\hat{x}}_k^p) \end{aligned}$$

and the matrix $\boldsymbol{C}_k^e$ by

$$\begin{aligned} \boldsymbol{C}_k^e &= s_k \boldsymbol{P}_k \ , \\ \boldsymbol{P}_k &= \boldsymbol{C}_k^p - \lambda_k \boldsymbol{C}_k^p (\boldsymbol{H}_k)^T \\ &\quad \left\{\boldsymbol{V}_k + \lambda_k \boldsymbol{H}_k \boldsymbol{C}_k^p (\boldsymbol{H}_k)^T\right\}^{-1} \boldsymbol{H}_k \boldsymbol{C}_k^p \ , \end{aligned}$$

where

$$\begin{aligned} s_k &= 1 + \lambda_k - \lambda_k (\underline{\hat{z}}_k - \boldsymbol{H}_k \underline{\hat{x}}_k^p)^T \\ &\quad \left\{\boldsymbol{V}_k + \lambda_k \boldsymbol{H}_k \boldsymbol{C}_k^p (\boldsymbol{H}_k)^T\right\}^{-1} (\underline{\hat{z}}_k - \boldsymbol{H}_k \underline{\hat{x}}_k^p) \ . \end{aligned} \quad (7)$$

The only assumption on the measurement uncertainties is that they are bounded according to $\underline{v}_k \in \boldsymbol{\mathcal{V}}_k$ with ellipsoidal sets $\boldsymbol{\mathcal{V}}_k$ given by

$$\boldsymbol{\mathcal{V}}_k = \{\underline{v}_k : \underline{v}_k^T (\boldsymbol{V}_k)^{-1} \underline{v}_k \leq 1\} \ . \quad (8)$$

$\boldsymbol{V}_k$ is the definition matrix of the set of all possible measurements according to (5), λ_k is a *fusion parameter* with $\lambda_k \in [0, \infty)$, and is chosen such that the volume of the bounding set $\boldsymbol{\mathcal{X}}_k^e$ is minimized. The calculation of the matrix $\boldsymbol{C}_k^e$ is very similar to the well–known Kalmanfilter equations. The additional parameter s_k according to (7) modifies the intermediate matrix $\boldsymbol{P}_k$ such that the resulting size of the bounding ellipsoidal set $\boldsymbol{\mathcal{X}}_k^e$ depends upon the actual measurement $\underline{\hat{z}}_k$, which is not the case for the Kalmanfilter.

B. Nonlinear Filtering Algorithm

In the proposed framework for nonlinear filtering, a standard linear filtering algorithm is applied in a higher dimensional space S^*. A transformed state $\underline{x}_k^*$ at time k in the L–dimensional space S^* is related to the state $\underline{x}_k$ in the original N–dimensional space S via a nonlinear transformation $\boldsymbol{T}(.)$ according to

$$\underline{x}_k^* = \boldsymbol{T}(\underline{x}_k) = \begin{bmatrix} T_1(\underline{x}_k) & T_2(\underline{x}_k) & \cdots & T_L(\underline{x}_k) \end{bmatrix}^T \ . \quad (9)$$

Hence, the transformation $\boldsymbol{T}(.)$ defines an N–dimensional manifold in the transformed space S^*, the so–called universal manifold U^*. The exact set of states $\boldsymbol{\mathcal{X}}_k$ in the original space S is then represented by an N–dimensional submanifold of U^*. This submanifold can now be bounded by the intersection of a simple shaped L–dimensional set *and* the universal manifold U^*. We define pseudo–ellipsoidal sets $\boldsymbol{\mathcal{X}}^*$ as sets, that are ellipsoidal in the transformed variables $\underline{x}^*$. An L–dimensional pseudo–ellipsoidal set according to

$$\boldsymbol{\mathcal{X}}_k^{e,*} = \{\underline{x}_k^* : (\underline{x}_k^* - \underline{\hat{x}}_k^{e,*})^T (\boldsymbol{C}_k^{e,*})^{-1} (\underline{x}_k^* - \underline{\hat{x}}_k^{e,*}) \leq 1\} \quad (10)$$

defines the resulting non–ellipsoidal set $\boldsymbol{\mathcal{X}}_k^e$ in the original, N–dimensional space S, that is used to bound the true set $\boldsymbol{\mathcal{X}}_k$. $\underline{\hat{x}}_k^{e,*}$ is the midpoint of the pseudo–ellipsoid and $\boldsymbol{C}_k^{e,*}$ is a symmetric, positive matrix. The advantage of this concept is, that it yields a simple description for complicated uncertainties $\boldsymbol{\mathcal{X}}_k^e$ in S and allows nonlinear recursive filtering in a system theoretic framework with little additional complexity compared to a linear filtering problem. The most complicated problem within this framework is to calculate characteristic values of the uncertainty set $\boldsymbol{\mathcal{X}}_k^e$ in S from $\boldsymbol{\mathcal{X}}_k^{e,*}$ via an inverse transformation.

In state estimation problems with unknown but bounded errors, a set of feasible states $\boldsymbol{\mathcal{X}}_k^e$ is determined based on observations $\underline{\hat{z}}_k$, that are related to the state $\underline{x}_k$ via a measurement equation

$$\underline{\hat{z}}_k = \underline{h}_k(\underline{x}_k) + \underline{v}_k \quad , \quad (11)$$

which is in general nonlinear. Similar to the linear case (5), $\underline{v}_k$ denotes additive, bounded measurement noise. To obtain the nonlinear transformation $\boldsymbol{T}(.)$ that defines the hyperspace S^* according to (9), a transformation $\underline{\eta}_k(.)$ is applied to both sides of the measurement equation (11), which results in

$$\underline{\eta}_k(\underline{\hat{z}}_k - \underline{v}_k) = \underline{\eta}_k(\underline{h}_k(\underline{x}_k)) \quad . \quad (12)$$

This transformation yields *nonlinear constraints* for the filtering result $\boldsymbol{\mathcal{X}}_k^e$, which are exploited to generate a tight approximation of the exact set $\boldsymbol{\mathcal{X}}_k$. In order to generate these constraints, the left hand side of equation (12) is exactly converted into

$$\underline{\eta}_k(\underline{\hat{z}}_k - \underline{v}_k) = \underline{\hat{z}}_k^* - \underline{v}_k^* \quad , \quad (13)$$

where $\underline{\hat{z}}_k^*$ represents the expected value of the transformed measurements in the transformed L–dimensional space S^*. $\underline{v}_k^* \in \boldsymbol{\mathcal{V}}_k^*$ accounts for the respective transformed measurement uncertainties. The right hand side of equation (12) can be expanded in the transformed space S^* into

$$\underline{\eta}_k(\underline{h}_k(\underline{x}_k)) = \boldsymbol{H}_k^* \underline{x}_k^* \quad . \quad (14)$$

(13) and (14) yield a pseudo–linear form of the measurement equation (11) transformed by $\underline{\eta}_k(.)$ resulting in

$$\underline{\hat{z}}_k^* = \boldsymbol{H}_k^* \underline{x}_k^* + \underline{v}_k^* \quad , \quad (15)$$

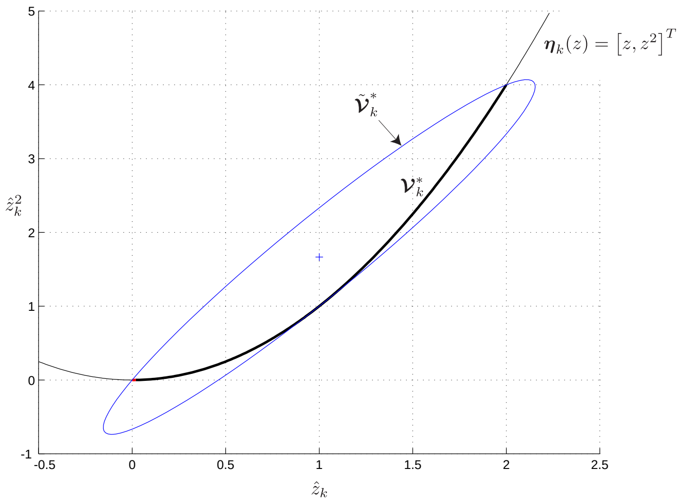

Fig. 2. Example for the application of a nonlinear transformation $\underline{\eta}_k(z) = [z, z^2]^T$ to generate additional nonlinear constraints for the estimated set of states $\boldsymbol{\mathcal{X}}_k^e$. The transformed, uncertain measurement $\underline{\hat{z}}_k^* = [\hat{z}_k, \hat{z}_k^2]^T$ is constrained to lie on a parabola $\boldsymbol{\mathcal{V}}_k^*$, that can be bounded by the ellipsoidal measurement uncertainty set $\tilde{\boldsymbol{\mathcal{V}}}_k^*$.

where $\boldsymbol{H}_k^*$ relates the transformed state vector *linear* to the measurements in the transformed space S^*.

Example: Consider the distance measurement equation

$$\hat{z}_k = x_k^2 + y_k^2 + v_k$$

with the two–dimensional state vector $\underline{x}_k = [x_k, y_k]^T$, bounded measurement noise $v_k \in [0, 2]$ and true system state $x_k = y_k = 0$. Applying the nonlinear transformation $\underline{\eta}_k(z) = [z, z^2]^T$ yields two nonlinear measurement equations

$$\begin{aligned} \hat{z}_k - v_k &= x_k^2 + y_k^2 \\ (\hat{z}_k - v_k)^2 &= x_k^4 + 2x_k^2 y_k^2 + y_k^4 \quad . \end{aligned}$$

These can be expanded according to (15) in a $L = 5$ dimensional space S^* with $\underline{x}_k^* = [x_k^2, y_k^2, x_k^2 y_k^2, x_k^4, y_k^4]^T$, two–dimensional measurement vector $\underline{\hat{z}}_k^* = [\hat{z}_k, \hat{z}_k^2]^T$ and related additive noise vector $\underline{v}_k^* = [v_k, 2\hat{z}_k v_k - v_k^2]^T$. The exact measurement uncertainty $\boldsymbol{\mathcal{V}}_k^*$ is a one–dimensional set, more precisely a segment of a parabola, in a two–dimensional measurement space. Figure 2 shows the parabola $\underline{\eta}_k(z) = [z, z^2]^T$ and the measurement uncertainty $\boldsymbol{\mathcal{V}}_k^*$, marked by a thick line. In order to apply the proposed nonlinear filter in the hyperspace S^*, a conservative ellipsoidal approximation $\tilde{\boldsymbol{\mathcal{V}}}_k^*$ for $\boldsymbol{\mathcal{V}}_k^*$ has to be found, that is depicted in Fig. 2. Note that there exist infinitely many valid, conservative approximations $\tilde{\boldsymbol{\mathcal{V}}}_k^*$ with $\boldsymbol{\mathcal{V}}_k^* \supset \tilde{\boldsymbol{\mathcal{V}}}_k^*$, the optimal one is the one for which the state estimate $\boldsymbol{\mathcal{X}}_k^e$ becomes least conservative.

The effect of the additional, nonlinear measurement equation, that generates a tighter estimated conservative

approximation $\boldsymbol{\mathcal{X}}_k^e$ for the exact set of states $\boldsymbol{\mathcal{X}}_k$ is demonstrated in the example for localization in the case of relative bearing measurements presented in Section V.
□

Using (15), the measurement update equations of the standard bounding ellipsoid filter (7) can directly be applied in the hyperspace S^* to calculate a bound for the complicated shaped estimated set of states $\boldsymbol{\mathcal{X}}_k^e$ in the original space S. This results in

$$\begin{aligned}\underline{\hat{x}}_k^{e,*} &= \underline{\hat{x}}_k^{p,*} + \lambda_k^* \boldsymbol{C}_k^{p,*} (\boldsymbol{H}_k^*)^T \\ &\quad \left\{ \boldsymbol{V}_k^* + \lambda_k^* \boldsymbol{H}_k^* \boldsymbol{C}_k^{p,*} (\boldsymbol{H}_k^*)^T \right\}^{-1} (\underline{\hat{z}}_k^* - \boldsymbol{H}_k^* \underline{\hat{x}}_k^{p,*})\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{C}_k^{e,*} &= s_k \boldsymbol{P}_k^* \quad , \qquad (16)\\ \boldsymbol{P}_k^* &= \boldsymbol{C}_k^{p,*} - \lambda_k^* \boldsymbol{C}_k^{p,*} (\boldsymbol{H}_k^*)^T \\ &\quad \left\{ \boldsymbol{V}_k^* + \lambda_k^* \boldsymbol{H}_k^* \boldsymbol{C}_k^{p,*} (\boldsymbol{H}_k^*)^T \right\}^{-1} \boldsymbol{H}_k^* \boldsymbol{C}_k^{p,*} \quad ,\end{aligned}$$

where

$$\begin{aligned}s_k &= 1 + \lambda_k^* - \lambda_k^* (\underline{\hat{z}}_k^* - \boldsymbol{H}_k^* \underline{\hat{x}}_k^{p,*})^T \\ &\quad \left\{ \boldsymbol{V}_k^* + \lambda_k^* \boldsymbol{H}_k^* \boldsymbol{C}_k^{p,*} (\boldsymbol{H}_k^*)^T \right\}^{-1} (\underline{\hat{z}}_k^* - \boldsymbol{H}_k^* \underline{\hat{x}}_k^{p,*}) \quad .\end{aligned}$$

The fusion parameter λ_k^* is selected such that the volume of the bounding set in the *original* space S is minimized. The resulting pseudo–ellipsoidal set $\boldsymbol{\mathcal{X}}_k^{e,*}$ is completely defined by its midpoint vector $\underline{\hat{x}}_k^{e,*}$ and the matrix $\boldsymbol{C}_k^{e,*}$, as introduced in (10).

Using this form to calculate the bounding pseudo–ellipsoidal set in the original space S possesses the following appealing properties:

- No additional uncertainty. The resulting set $\boldsymbol{\mathcal{X}}_k^e$ is a subset of the union of the predicted set $\boldsymbol{\mathcal{X}}_k^P$ and the measurement set $\boldsymbol{\mathcal{X}}_k^M$.
- Conservativeness. If the true set $\boldsymbol{\mathcal{X}}_k$ is contained in $\boldsymbol{\mathcal{X}}_k^P$, the resulting set $\boldsymbol{\mathcal{X}}_k^e$ is a conservative approximation for $\boldsymbol{\mathcal{X}}_k$.
- Tighter bound. The resulting nonlinear, implicit polynomial bounding set is a better approximation for the exact set of states $\boldsymbol{\mathcal{X}}_k$ than a standard ellipsoidal bounding set obtained by linearization of the original, nonlinear problem or a box–shaped set. This means, it is a conservative bound with a volume smaller than the volume of the ellipsoidal bounding set or box–shaped sets.

IV. Transformation of the localization problem

To apply the described framework for nonlinear filtering to the problem of localization in the case of relative bearing measurements, each measurement equation (2) for two landmarks L_i, L_j is first transformed into a nonlinear measurement equation according to

$$(x_k - x_M(\hat{\gamma}_{i,j}))^2 + (y_k - y_M(\hat{\gamma}_{i,j}))^2 = r^2(\hat{\gamma}_{i,j}) \quad , \qquad (17)$$

where $\hat{\gamma}_{i,j} = \gamma_{i,j} + v^{i,j}$ is the observed uncertain difference angle and time index k has been omitted for brevity. (17) can be derived from (2) exploiting the fact, that all robot positions, from which a constant difference angle is observed, lie on a circular arc $\boldsymbol{\mathcal{C}}_k^{i,j}$ [4], [10]. Hence, (17) is the definition of the circle $\bar{\boldsymbol{\mathcal{C}}}_k^{i,j}$ corresponding to $\boldsymbol{\mathcal{C}}_k^{i,j}$ whose midpoint $\underline{M}(\hat{\gamma}_{i,j}) = [x_M, y_M]^T$ and radius $r(\hat{\gamma}_{i,j})$ are a function of the observed difference angle $\hat{\gamma}_{i,j}$ given by

$$\underline{M}(\hat{\gamma}_{i,j}) = \underline{x}_{LM_i} + \frac{1}{2}\left(\underline{x}_{LM_j} - \underline{x}_{LM_i}\right) - \frac{1}{2}\cot(\hat{\gamma}_{i,j})\underline{\Delta}_{i,j}^{\perp} \qquad (18)$$

and

$$r(\hat{\gamma}_{i,j}) = \frac{\|\underline{x}_{LM_j} - \underline{x}_{LM_i}\|}{|2\sin(\hat{\gamma}_{i,j})|} \quad , \qquad (19)$$

where $\underline{x}_{LM_i}$ is the position vector of landmark i and $\underline{\Delta}_{i,j}^{\perp}$ is the vector orthogonal to $\underline{x}_{LM_j} - \underline{x}_{LM_i}$ with $\|\underline{\Delta}_{i,j}^{\perp}\| = \|\underline{x}_{LM_j} - \underline{x}_{LM_i}\|$. The example given in Fig. 1 shows the set $\boldsymbol{\mathcal{M}}_k^{1,2}$ resulting from the union of all circular arcs $\boldsymbol{\mathcal{C}}_k^{i,j}$ with $\hat{\gamma}_{1,2} \in \left[\gamma_{1,2}(k) - \epsilon_{1,2,k}^v, \gamma_{1,2}(k) + \epsilon_{1,2,k}^v\right]$. Note that the additional linear constraint $y_k < 1$ is required to select the subset of points of the circles $\bar{\boldsymbol{\mathcal{C}}}_k^{1,2}$ (17) that are part of the circular arcs $\boldsymbol{\mathcal{C}}_k^{1,2}$. The nonlinear measurement equation (17) can now be transformed into two nonlinear measurement equations. Each of the two measurement equations describes a distance measurement. This transformation is possible for the assumed bounded error model because the intersection of the sets defined by the two measurements yields exactly the original, crescent shaped measurement set $\boldsymbol{\mathcal{M}}_k^{i,j}$ resulting from the uncertain relative bearing measurement (17). The measurement equations can be written in state–space form according to (11) where

$$\underline{h}_k(\underline{x}_k) = \begin{bmatrix}(x_k - x_{M_1})^2 + (y_k - y_{M_1})^2 \\ (x_k - x_{M_2})^2 + (y_k - y_{M_2})^2\end{bmatrix} \qquad (20)$$

and

$$\underline{\hat{z}}_k = \begin{bmatrix}(r_1 + \frac{d}{2})^2 \\ (r_2 - \frac{d}{2})^2\end{bmatrix} \quad . \qquad (21)$$

$\underline{M}_l = [x_{M_l}, y_{M_l}]^T$, $l = 1,2$ are the midpoints of the outer ring R_1 and the inner ring R_2 containing the exact measurement set $\boldsymbol{\mathcal{M}}_k^{i,j}$, and $r_l = r_l^k$ are the related radii at time step k, calculated from (18), (19) when the bounds of the admissible interval for $\hat{\gamma}_{i,j}$ are inserted. The parameter d corresponds to the uncertainty of the transformed distance measurements, and can be directly obtained from $\underline{M}_l$ and r_l, $l = 1,2$. For the example given in Fig. 1, a transformation into two distance measurement equations according to (11) is graphically illustrated in Fig. 3. Each transformed measurement equations corresponds to a ring with midpoint $\underline{M}_1$ respectively $\underline{M}_2$ and radius $r_1 + \frac{d}{2}$ and $r_2 - \frac{d}{2}$. It can be seen, that the intersection of the two rings is equivalent to the original measurement set $\boldsymbol{\mathcal{M}}_k^{1,2}$ depicted in Fig. 1.

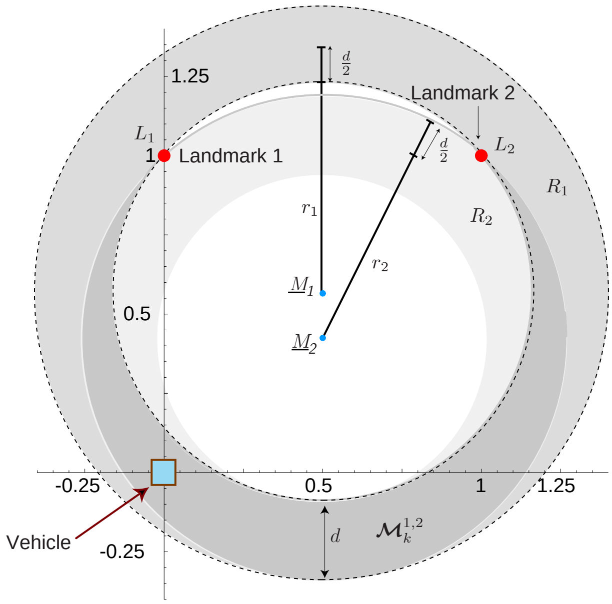

Fig. 3. Localization by two angle measurements with bounded uncertainties. Graphical illustration of the transformation of the measurement equation. Each grey shaded ring R_1 and R_2 corresponds to a transformed, pseudo–linear measurement equation. The intersection of R_1 and R_2 is equivalent to the desired set of circular arcs $\mathcal{M}_k^{i,j}$.

Expanding these equations into an intermediate pseudo–linear form in an $L = 4$ dimensional hyperspace $\bar{S}^*$ yields

$$\bar{\underline{z}}_k^* = \bar{\boldsymbol{H}}_k^* \bar{\underline{x}}_k^* + \bar{\underline{v}}_k^* \quad , \tag{22}$$

with the new state vector $\bar{\underline{x}}_k^* = \bar{\boldsymbol{T}}(\underline{x}_k)$, measurement matrix $\bar{\boldsymbol{H}}_k^*$, and measurement vector $\bar{\underline{z}}_k^*$ according to

$$\bar{\underline{x}}_k^* = \bar{\boldsymbol{T}}(\underline{x}_k) = \left[x_k, y_k, x_k^2, y_k^2\right]^T$$
$$\bar{\boldsymbol{H}}_k^* = \begin{bmatrix} -2x_{M_1}, -2y_{M_1}, 1, 1 \\ -2x_{M_2}, -2y_{M_2}, 1, 1 \end{bmatrix}$$
$$\bar{\underline{z}}_k^* = \begin{bmatrix} (r_1 + \frac{d}{2})^2 - (x_{M_1}^2 + y_{M_1}^2) \\ (r_2 - \frac{d}{2})^2 - (x_{M_2}^2 + y_{M_2}^2) \end{bmatrix} \quad ,$$

where the bar denotes variables related to the intermediate, pseudo–linear form. The corresponding bounded measurement noise $\bar{\underline{v}}_k^* = \left[\bar{v}_k^{*,1}, \bar{v}_k^{*,2}\right]^T$ is constrained by the intervals

$$\bar{v}_k^{*,1} \in \left[-((d/2)^2 + d\, r_1), 3/4\, d^2 + d\, r_1\right] \quad ,$$
$$\bar{v}_k^{*,2} \in \left[-((d/2)^2 + d\, r_2), 3/4\, d^2 + d\, r_2\right] \quad ,$$

where $d = d_k^{i,j}$ corresponds to the "thickness" of the rings belonging to $\boldsymbol{\mathcal{X}}_k^M$ (see Fig 3). The pseudo–linear measurement equation (22) can directly be used for filtering with the described pseudo–ellipsoidal approach, as will be demonstrated in an example in Section V.

Application of the nonlinear transformation $\underline{\eta}_k(.)$ according to (12) generates an even tighter approximation of the exact measurement set $\boldsymbol{\mathcal{X}}_k^M$. Here the transformation $\underline{\eta}_k(z) = \left[z, z^2\right]^T$ is chosen to obtain the final pseudo–linear measurement equation. Transformations of higher order yield increasingly tighter approximations, but require higher computational effort. This transformation is applied to each row of (22) separately, which results in two measurement equations with the new state vector $\underline{x}_k^*$ given by

$$\underline{x}_k^* = [\, x_k, y_k, x_k y_k, x_k^2, y_k^2, \; x_k^3 + x_k y_k^2, x_k^2 y_k + y_k^3, x_k^4 + 2x_k^2 y_k^2 + y_k^4 \,]^T \tag{23}$$

and measurement matrices ${}^i\boldsymbol{H}_k^*$ with

$${}^i\boldsymbol{H}_k^* = \begin{bmatrix} -2x_{M_i} & -4x_{M_i}^3 - 4x_{M_i} y_{M_i}^2 \\ -2y_{M_i} & -4y_{M_i}^3 - 4x_{M_i}^2 y_{M_i} \\ 0 & 8x_{M_i} y_{M_i} \\ 1 & 6x_{M_i}^2 + 2y_{M_i}^2 \\ 1 & 2x_{M_i}^2 + 6y_{M_i}^2 \\ 0 & -4x_{M_i} \\ 0 & -4y_{M_i} \\ 0 & 1 \end{bmatrix}^T \quad i = 1,2 \tag{24}$$

in the $L = 8$ dimensional space S^*. The associated measurement vectors ${}^i\underline{\hat{z}}_k^*$ are given by

$${}^1\underline{\hat{z}}_k^* = \begin{bmatrix} (r_1 + \frac{d}{2})^2 - (x_{M_1}^2 + y_{M_1}^2) \\ (r_1 + \frac{d}{2})^4 - (x_{M_1}^4 + 2x_{M_1}^2 y_{M_1}^2 + y_{M_1}^4) \end{bmatrix} \tag{25}$$

and

$${}^2\underline{\hat{z}}_k^* = \begin{bmatrix} (r_2 - \frac{d}{2})^2 - (x_{M_2}^2 + y_{M_2}^2) \\ (r_2 - \frac{d}{2})^4 - (x_{M_2}^4 + 2x_{M_2}^2 y_{M_2}^2 + y_{M_2}^4) \end{bmatrix} \quad . \tag{26}$$

The measurement uncertainty sets ${}^i\tilde{\boldsymbol{\mathcal{V}}}_k^*, i = 1,2$ with associated definition matrices ${}^i\boldsymbol{V}_k^*$ are given by

$${}^i\tilde{\boldsymbol{\mathcal{V}}}_k^* = \{\underline{v}_k^* : [\underline{v}_k^* - {}^i\underline{\hat{v}}_k^*]^T ({}^i\boldsymbol{V}_k^*)^{-1} [\underline{v}_k^* - {}^i\underline{\hat{v}}_k^*] \le 1\}$$

for the transformed measurement equations (15). Similar to the example in Section III-B, their parameters given by the midpoint vectors ${}^i\hat{\underline{v}}^*$ and the matrices ${}^i\boldsymbol{V}_k^*$ are calculated such that ${}^i\hat{\underline{z}}_k^* - {}^i\boldsymbol{H}_k^*\underline{x}_k^* \in {}^i\tilde{\boldsymbol{\mathcal{V}}}_k^*$ for all possible $\underline{v}_k^*$. The calculation of an arbitrary conservative pseudo–ellipsoidal set ${}^i\tilde{\boldsymbol{\mathcal{V}}}_k^*$ is simple, yet to find the minimum volume ellipsoidal set ${}^i\tilde{\boldsymbol{\mathcal{V}}}_k^*$ is in general not a trivial task. A closed–form solution is available and has been used for the transformation $\underline{\eta}_k(z) = \left[z, z^2\right]^T$, as demonstrated in the example in Section III-B. For application of the filtering algorithm (16) $\hat{\underline{v}}_k^*$ is subtracted from $\hat{\underline{z}}_k^*$ to obtain zero mean measurement noise.

V. Solution of the Localization problem using nonlinear filtering

To demonstrate the performance of the proposed approach for relative bearing localization, consider the typical scenario depicted in Fig. 4. A vehicle V at position $\underline{x}_k = [0,0]^T$ conducts bearing measurements to $N = 3$ landmarks located at $L_1 = [0,1]^T$, $L_2 = [1,1]^T$ and $L_3 = [-1,0]^T$. According to the notation presented in the problem statement, these measurements define 3 circular arcs $\boldsymbol{\mathcal{C}}_k^{i,j}$ and associated measurement sets $\boldsymbol{\mathcal{M}}_k^{i,j}$. Note, that the true position of the vehicle is contained in the intersection of all 3 measurement sets.

To solve the given localization problem shown in Fig. 4, a noninformative prior $\boldsymbol{\mathcal{X}}_k^P$ is assumed, i. e., no prior knowledge about the initial position of the robot is given, and $\boldsymbol{\mathcal{X}}_k^P \cap \boldsymbol{\mathcal{X}}_k^M = \boldsymbol{\mathcal{X}}_k^M$. This means, that position estimation is based on the measurements only. Consecutive application of the update equations (7) for each pair (i,j) of landmarks L_i, L_j yields the desired estimates $\hat{\underline{x}}_k^e$ of the vehicle position together with the associated sets $\boldsymbol{\mathcal{X}}_k^e$ of all vehicle positions compatible with the given measurements. To calculate these desired quantities, the pseudo–ellipsoidal bounding set $\boldsymbol{\mathcal{X}}_k^{e,*}$ according to (10) is evaluated on the two–dimensional manifold U^* defined in an 8–dimensional space by the nonlinear transformation $\boldsymbol{T}(.)$ given by (23). This yields a higher order implicit polynomial description of the uncertainty set $\boldsymbol{\mathcal{X}}_k^e$ in the original space S. From this implicit description, characteristic values of $\boldsymbol{\mathcal{X}}_k^e$ like the boundary of the set are calculated numerically by means of the inverse transformation.

First, the intermediate measurement equations in pseudo–linear form according to (22) are directly used with the proposed new filter algorithm, which is equivalent to choosing $\underline{\eta}_k(.)$ as the identity transformation $\underline{\eta}_k(z) = z$. This means, an $L = 4$ dimensional hyperspace S^* is used to describe the $N = 2$ dimensional uncertainty set $\boldsymbol{\mathcal{X}}_k^M$ in the original space S. The resulting exact measurement set $\boldsymbol{\mathcal{M}}^{3,1}$ after including the first relative bearing measurement between landmark L_3 and L_1 is shown in Fig. 4 as a shaded crescent shaped area. The related approximation $\tilde{\boldsymbol{\mathcal{M}}}^{3,1}$ calculated with the proposed filter algorithm is marked by a thick black line in Fig. 6b).

Then, the additional transformation $\underline{\eta}_k(z) = \left[z, z^2\right]^T$ is applied yielding measurement equation (16) with $\underline{x}_k^*$, $\boldsymbol{H}_k^*$ and $\hat{\underline{z}}_k^*$ given by (23), (24) and (25),(26) in an $L = 8$ dimensional space. Fig. 5 and the zoomed clipping depicted in Fig. 6a) show the result obtained after including the first relative bearing measurement between landmark L_3 and L_1. It can clearly be seen, that the shaded exact measurement set $\boldsymbol{\mathcal{M}}^{3,1}$ in Fig. 4 is very tightly approximated by the proposed, implicit polynomial bound resulting from the nonlinear pseudo–ellipsoidal filter, which is marked by a thick black line in Fig. 5. Note that this approximation obtained with the additional transformation $\underline{\eta}_k(.)$ is even better than the result obtained by straightforward application of the expanded measurement equation (22), which is depicted in Fig. 6b). Figure 7 shows the final result after all measurements have been included. From the zoomed clipping shown in Fig. 8 it can be appreciated, that all $N = 3$ measurement sets $\boldsymbol{\mathcal{M}}_k^{i,j}$ have been taken into account and a tight upper bound for the exact set has been achieved. For reference purposes, the tightest possible axis aligned box–shaped set $\boldsymbol{\mathcal{X}}_k^{M,B}$ is also shown, which is obviously a much more conservative approximation.

VI. Conclusion and Future Work

In this article a new approach for localization of a mobile robot for the case of relative bearing measurements has been described. The key idea of the new approach is to transform the given problem to a form amenable to the application of a system theoretic filtering framework for nonlinear bounded error estimation. For that purpose, an existing approach for nonlinear filtering based on overparametrization was generalized and yields a simple, intuitive description of the nonlinear filtering problem. A complicated uncertainty $\boldsymbol{\mathcal{X}}_k$ in the original space $S = \mathbb{R}^2$ is represented by a simpler shaped uncertainty $\boldsymbol{\mathcal{X}}_k^*$ and an associated transformation $\boldsymbol{T}(.)$ in a higher dimensional hyperspace S^*. This approach allows to apply well–known, linear ellipsoidal bounding filters in the higher dimensional space S^* to obtain a tight approximation of the desired uncertainty $\boldsymbol{\mathcal{X}}_k$, that describes the position of the robot. Nonlinear constraints for the estimated position in the original space S are generated by a nonlinear transformation $\underline{\eta}_k(.)$, which is applied to the measurement equation. The most challenging problem in terms of computational complexity is the determination of characteristic values of the complicated uncertainty $\boldsymbol{\mathcal{X}}_k$ by means of an inverse transformation, which in general requires numerical calculations. In a practical localization example, it was shown, that a second order transformation with $\underline{\eta}_k(z) = \left[z, z^2\right]^T$ yields an approximation $\boldsymbol{\mathcal{X}}_k^e$, that is very close to optimal and outperforms simple, approximation schemes based on

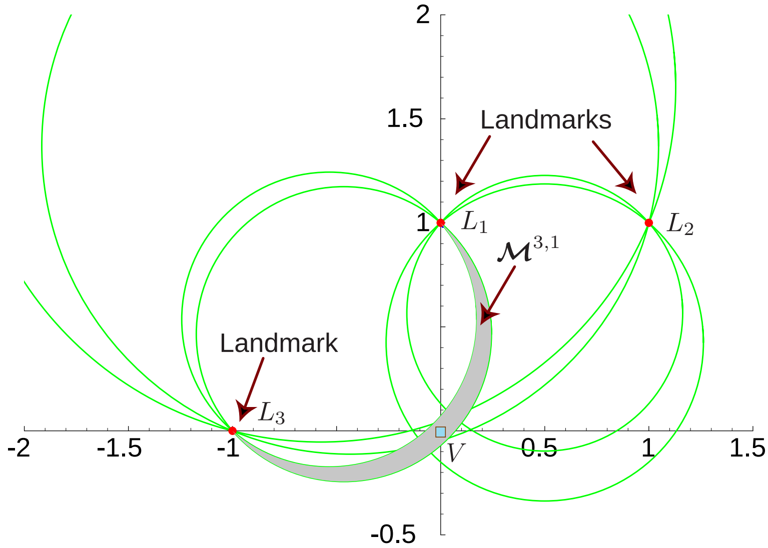

Fig. 4. Three angle measurements with bounded uncertainties: The shaded set is the resulting, exact measurement set $\mathcal{M}^{3,1}$ after the first filtering step.

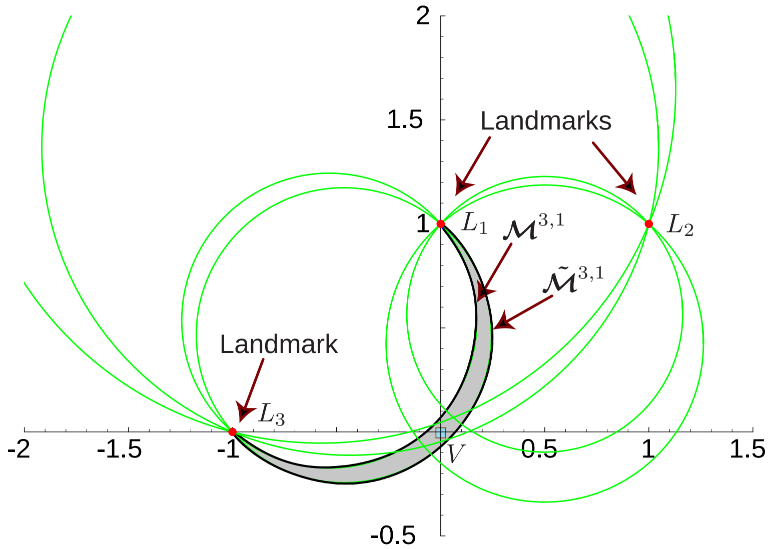

Fig. 5. Three angle measurements with bounded uncertainties: The shaded resulting, exact measurement set $\mathcal{M}^{3,1}$ and the implicit polynomial approximation $\tilde{\mathcal{M}}^{3,1}$ (thick black line) after the first filtering step with the new nonlinear Pseudo–Ellipsoidal Filter. The nonlinear transformation $\underline{\eta}_k(z) = [z, z^2]^T$ was applied to generate additional constraints.

axis–aligned boxes. Increasing the order of the polynomial transformation $\underline{\eta}_k(z)$ results in even better approximations, but requires additional computational effort.

Future research on localization using nonlinear filtering will focus on the question, how to include nonlinear prediction of the resulting feasible set of states and how system noise can be incorporated in the proposed approach.

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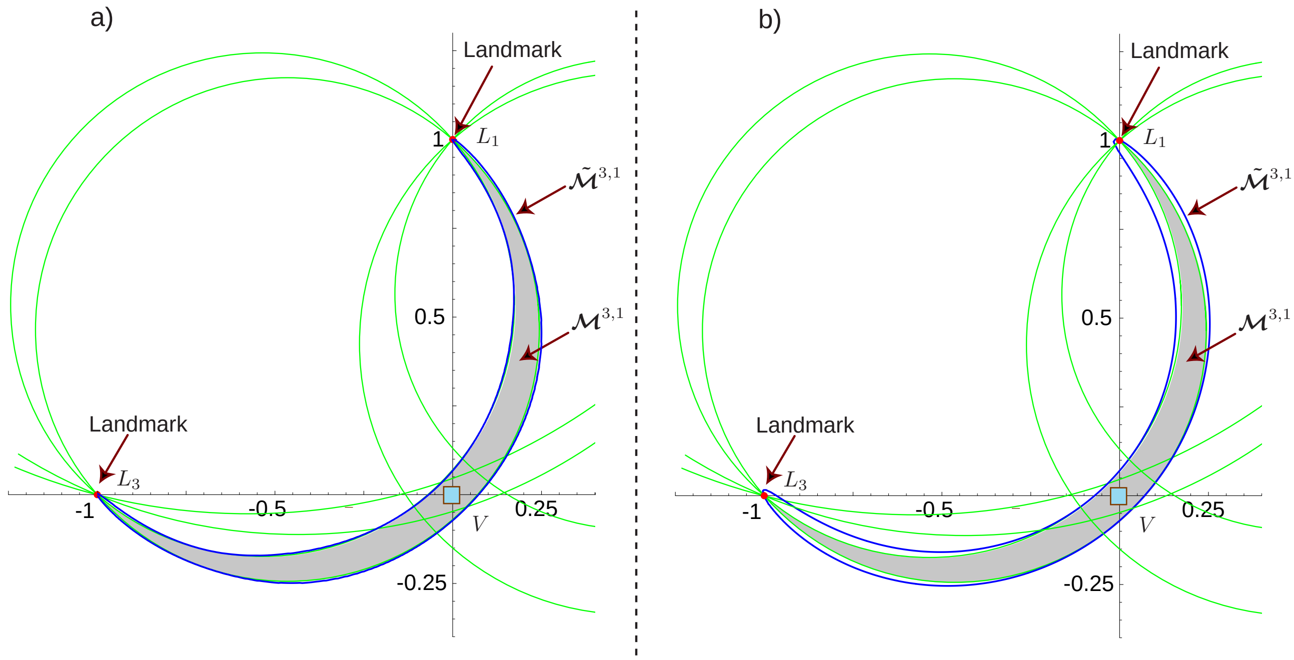

Fig. 6. Three angle measurements with bounded uncertainties: The shaded resulting, exact measurement set $\boldsymbol{\mathcal{M}}^{3,1}$ and the implicit polynomial approximation $\tilde{\boldsymbol{\mathcal{M}}}^{3,1}$ (thick black line) after the first filtering step with the new nonlinear Pseudo–Ellipsoidal Filter. a) Zoom from Fig. 5: The nonlinear transformation $\underline{\eta}_k(z) = \left[z, z^2\right]^T$ was applied to generate additional constraints. b) *No* additional nonlinear transformation $\underline{\eta}_k(.)$ was applied. Please note: The resulting approximation $\boldsymbol{\mathcal{M}}^{3,1}$ is more conservative than the one obtained in Fig. 5, but obviously still much tighter than the smallest possible box–shaped approximation using a single box.

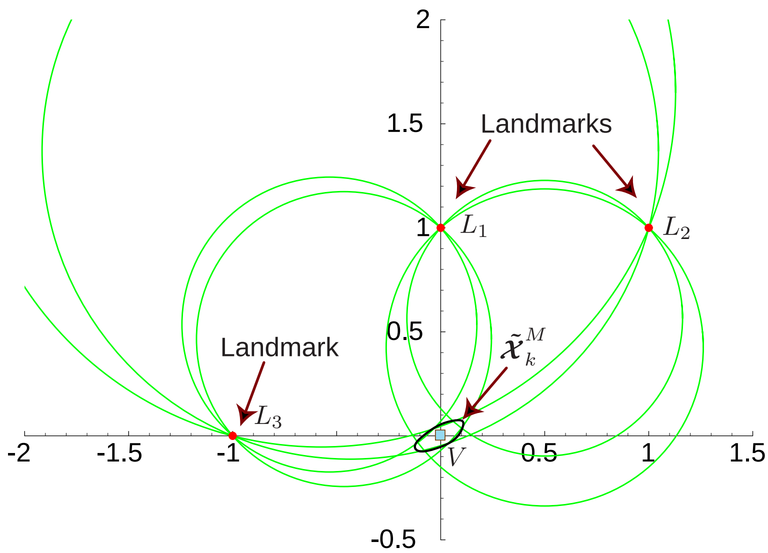

Fig. 7. Three angle measurements with bounded uncertainties: Resulting exact measurement set $\boldsymbol{\mathcal{X}}_k^M$ and implicit polynomial approximation $\tilde{\boldsymbol{\mathcal{X}}}_k^M$ resulting from the new nonlinear Pseudo–Ellipsoidal Filter. The nonlinear transformation $\underline{\eta}_k(z) = \left[z, z^2\right]^T$ was applied to generate additional constraints.

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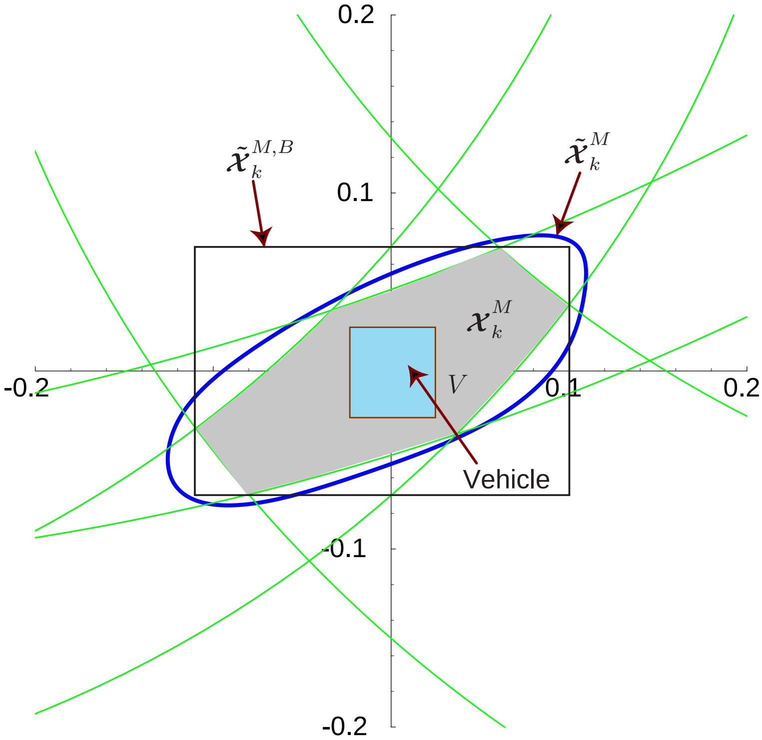

Fig. 8. Zoomed clipping from Fig. 7: Three angle measurements with bounded uncertainties: Resulting exact measurement set $\boldsymbol{\mathcal{X}}_k^M$ and implicit polynomial approximation $\tilde{\boldsymbol{\mathcal{X}}}_k^M$ resulting from the new nonlinear Pseudo–Ellipsoidal Filter. The nonlinear transformation $\underline{\eta}_k(z) = [z, z^2]^T$ was applied to generate additional constraints. $\tilde{\boldsymbol{\mathcal{X}}}_k^{M,B}$ is the tightest possible axis aligned box–shaped set, which is much more conservative. Please note: The shaded exact set $\boldsymbol{\mathcal{X}}_k^M$ cannot exactly be represented by polytopes.

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