Wavefront Orientation Estimation Based on Progressive Bingham Filtering

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***Abstract*—In this paper, we propose the Progressive Bingham Filter (PBF), a novel stochastic filtering algorithm for nonlinear spatial orientation estimation. As an extension of the orientation filter previously proposed only for the identity measurement model based on the Bingham distribution, our method is able to handle arbitrary measurement models. Instead of the sampling-approximation scheme used in the prediction step, a closed-form solution is possible when the system equation is based on the Hamilton product. Besides stochastic approaches, we also introduce the Spherical Averaging Method (SAM), which is an application of the Riemannian averaging technique. The two approaches are then applied to a specific problem where the wavefront orientation is estimated based on Time Differences of Arrival (TDOA) and evaluated in simulations. The results show theoretical competitiveness of the PBF.**

Keywords*—*Nonlinear Progressive Filtering, Directional Estimation, Bingham Distribution, Time Difference of Arrival

I. Introduction

Spatial orientation estimation plays an important role in various application scenarios including multilateration, computer vision [1], robotic manipulation and navigation as well as perception [2]–[4]. Over the years, there have been extensive efforts towards robust and accurate estimation techniques for rotations or more general spatial transformations, among which the stochastic approaches are quite popular. However, mostly due to the nonlinear group structure of the special orthogonal group SO(3), conventional stochastic filtering approaches cannot be trivially applied [5], [6]. More specifically, the following issues can make the stochastic estimation of orientations challenging.

First, different representation methods of spatial orientation, which determine the state domains, can have a significant impact on the problem formulation. One of the mostly used representations is the Euler angle parameterization. However, Euler angles have the problem of gimbal lock [2] and also ambiguity [7]. Another choice is to use the matrix of the 3-D rotation group, which can cause numerical instability due to over-parametrization. In contrast, the unit quaternion is more intuitive to compose and involves less redundancy. Second, spatial orientations are periodic and belong to the SO(3), whose group structure is nonlinear. Some stochastic filters have been proposed to tackle the nonlinear estimation problem in a locally linearized domain, e.g., the well-known Extended Kalman Filter (EKF) [1] and Unscented Kalman Filter (UKF) [8]. However, this could be problematic for high noise level and fast orientation changes. Moreover, they also lack stochastic modeling of uncertain orientations directly on the nonlinear manifold itself but rather assume Gaussian-distributed noise on the locally linearized domain.

Directional statistics [9] specifically models uncertain directional variables, e.g., angles or orientations, directly on their periodic and nonlinear domain. In [10]–[13], stochastic filters using specific directional distributions, e.g., von Mises-Fisher distribution, (bivariate) wrapped normal distribution, and Bingham distribution, have been proposed. They typically rely on a deterministic sampling scheme analogous to the unscented transform for propagation through nonlinear system dynamics. The Bingham distribution [14], which can be seen as a modified version of Gaussian distribution on the sphere or hypersphere, has shown benefits for nonlinear filtering based on a unit quaternion representation. However, the previously proposed Bingham Filter (BF) [15] is only applicable to the identity measurement model, which limits its application in most of the real-world scenarios. Moreover, the sampling-approximation scheme during the prediction step might be time-consuming.

In this work, we introduce the Progressive Bingham Filter (PBF), an extended version of the originally proposed Bingham Filter (BF) based on unit quaternion representation, to handle arbitrary nonlinear measurement models. For the prediction step with system equations purely based on the Hamilton product, we give a closed-form solution as an alternative to the deterministic sampling-based approach. Besides the stochastic approaches through unit quaternion filtering, we apply an existing Riemannian averaging technique [16] directly on the manifold $\mathbb{S}^2$ of spatial orientations. The evaluation is simulation-based with an application to orientation estimation using noisy Time Difference of Arrival (TDOA) measurements, which is of broad interest in signal processing and surveillance. The results show that the proposed PBF outperforms the original PF and the Riemannian averaging technique under the system noise assumed to be Bingham-distributed.

The remaining parts of this paper are structured as follows. In Sec. II, preliminaries about unit quaternions and the Bingham distribution are introduced. The closed-form solution for Hamilton product-based system equations is explained afterwards. Sec. III gives a detailed introduction to the PBF for arbitrary measurement models. In Sec. IV, a non-stochastic technique, the Spherical Averaging Method (SAM), is introduced. The evaluation is performed in Sec. V for orientation estimation based on Time Differences of Arrival (TDOA). Finally, the work is concluded in Sec. VI.

II. Preliminaries

A. Unit Quaternions and Spatial Orientations

In this part, we first give a basic introduction to the unit quaternions for orientation representation and their manifold structure. Some detailed preliminaries for quaternion arithmetics can be found in [17]–[19]. Second, we recap the definition of the Bingham Distribution and propose a novel closed-form solution for its propagation given a purely Hamilton product-based system equation.

The rotation representation using the unit quaternions is essentially a re-parameterization of the axis-angle method. Given a rotation axis $\mathbf{u} \in \mathbb{S}^2$ and rotation angle θ, the corresponding unit quaternion can then be composed as

$$\mathbf{q} = \cos\left(\frac{\theta}{2}\right) + \mathbf{u}\sin\left(\frac{\theta}{2}\right) . \tag{1}$$

For an arbitrary vector $\mathbf{v} \in \mathbb{R}^3$, it can be rotated accordingly as follows

$$\mathbf{v}' = \mathbf{q} \otimes \mathbf{v} \otimes \mathbf{q}^* , \tag{2}$$

with $\mathbf{q}^* = \mathrm{diag}(1, -1, -1, -1)\,\mathbf{q}$ denoting the conjugate of the quaternion $\mathbf{q}$ and $\otimes$ the Hamilton product [17]. Note the norm of the quaternion, $\sqrt{\mathbf{q} \otimes \mathbf{q}^*}$, gives the same value as its Euclidean norm, thus the unit quaternions essentially form a hypersphere in the four-dimensional Euclidean space, namely $\mathbf{q} \in \mathbb{S}^3 \subset \mathbb{R}^4$. It can be easily verified that the quaternion defined in (1) is a unit one. The Hamilton product of two quaternions can also be written as matrix-vector product [18], namely

$$\mathbf{p} \otimes \mathbf{q} = \mathbf{Q}_p^l \mathbf{q} = \mathbf{Q}_q^r \mathbf{p} , \tag{3}$$

with

$$\begin{aligned} \mathbf{Q}_p^l &= \begin{pmatrix} p_1 & -p_2 & -p_3 & -p_4 \\ p_2 & p_1 & -p_4 & p_3 \\ p_3 & p_4 & p_1 & -p_2 \\ p_4 & -p_3 & p_2 & p_1 \end{pmatrix} , \\ \mathbf{Q}_q^r &= \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & q_4 & -q_3 \\ q_3 & -q_4 & q_1 & q_2 \\ q_4 & q_3 & -q_2 & q_1 \end{pmatrix} . \end{aligned} \tag{4}$$

Here, the matrix elements are directly taken from the quaternion vectors, e.g., $\mathbf{q} = [q_1, q_2, q_3, q_4]^\top$. Furthermore, $\forall\, \mathbf{q} \in \mathbb{S}^3$, it can be proven that its matrix representation $\mathbf{Q}_q^l, \mathbf{Q}_q^r \in$ SO(4), which is the 4-dimensional rotation group. For unit quaternions, the inverse, which performs rotation in the opposite direction, equals to the conjugate, namely $\mathbf{q}^{-1} = \mathbf{q}^*$. Thus, for an inverted unit quaternion $\mathbf{q}^{-1}$, its matrix representation is then the transpose of the original one, i.e., $\mathbf{Q}_{q^{-1}}^l = (\mathbf{Q}_q^l)^\top$ and $\mathbf{Q}_{q^{-1}}^r = (\mathbf{Q}_q^r)^\top$.

B. Bingham Distribution under Hamilton Product

As (2) indicates, two antipodal unit quaternions on $\mathbb{S}^3$, e.g., $\mathbf{q}$ and $-\mathbf{q}$, represent the same spatial orientation. In order to model uncertain unit quaternions, the distribution should thus be antipodally symmetric. The Bingham Distribution (BD) on the hypersphere $\mathbb{S}^{d-1}$ is generated by restricting a Gaussian distribution in $\mathbb{R}_d$ on the unit hypersphere. More specifically, the BD on $\mathbb{S}^3$ for modeling uncertain unit quaternions is defined as

$$\begin{aligned} f(\mathbf{q}) &= \mathcal{B}(\mathbf{q}; \mathbf{M}, \mathbf{Z}) \\ &= \frac{1}{N(\mathbf{Z})} \exp(\mathbf{q}^\top \mathbf{M}\mathbf{Z}\mathbf{M}^\top \mathbf{q}), \quad \mathbf{q} \in \mathbb{S}^3 , \end{aligned} \tag{5}$$

where the diagonal matrix $\mathbf{Z} = \mathrm{diag}(z_1, z_2, z_3, z_4)$ indicates the concentration with increasing entries $z_1 \leq z_2 \leq z_3 \leq z_4 \leq 0$, $\mathbf{M}$ is an orthonormal matrix, and $N(\mathbf{Z})$ is the normalization constant which only depends on the concentration matrix. Given a unit quaternion that is Bingham-distributed, i.e., $\mathbf{q}_k \sim \mathcal{B}(\mathbf{M}_k, \mathbf{Z}_k)$, the transformed unit quaternion under the system equation $\mathbf{q}_{k+1} = \mathbf{q}_k \otimes \mathbf{u}_k$ (without noise) is still Bingham-distributed. The proof is as follows. It can be derived that $\mathbf{q}_k = \mathbf{q}_{k+1} \otimes \mathbf{u}_k^{-1} = (\mathbf{Q}_u^r)^\top \mathbf{q}_{k+1}$, thus

$$\begin{aligned} & f(\mathbf{q}_{k+1}) \\ &= f(\mathbf{q}_{k+1} \otimes \mathbf{u}_k^{-1}; \mathbf{M}_k, \mathbf{Z}_k) \\ &= \frac{1}{N(\mathbf{Z}_k)} \exp\left(((\mathbf{Q}_u^r)^\top \mathbf{q}_{k+1})^\top \mathbf{M}_k \mathbf{Z}_k \mathbf{M}_k^\top (\mathbf{Q}_u^r)^\top \mathbf{q}_{k+1}\right) \\ &= \frac{1}{N(\mathbf{Z}_k)} \exp\left(\mathbf{q}_{k+1}^\top (\mathbf{Q}_u^r \mathbf{M}_k) \mathbf{Z}_k (\mathbf{Q}_u^r \mathbf{M}_k)^\top \mathbf{q}_{k+1}\right) . \end{aligned} \tag{6}$$

Since $\mathbf{Q}_u^r \in$ SO(4), $\mathbf{Q}_u^r \mathbf{M}_k$ is still an orthonormal matrix. Moreover, the normalization constant depends only on the concentration matrix $\mathbf{Z}$. Thus, the transformed unit quaternions are still Bingham-distributed with the same concentration rotated on $\mathbb{S}^3$ according to the system input $\mathbf{u}_k$, namely $\mathbf{q}_{k+1} \sim \mathcal{B}(\mathbf{Q}_u^r \mathbf{M}_k, \mathbf{Z}_k)$. Compared to the sampling-approximation scheme proposed previously [15], this should give some speed-up during the prediction step for stochastic estimation under a system dynamics purely based on the Hamilton product.

III. Progressive Bingham Filter

In this section, the novel Progressive Bingham Filter (PBF) is introduced. Unlike the previously proposed Bingham Filter (BF) [15] where an identity measurement model is assumed, it can handle arbitrary measurement models by applying a progressive update step. Moreover, by using the closed-form solution introduced in II-B, our method shows better efficiency with the system dynamics purely conducting the Hamilton product.

A. Prediction Step

The system dynamics is assumed to be

$$\mathbf{q}_k = a(\mathbf{q}_{k-1}, \mathbf{u}_k) \otimes \mathbf{w}_k , \tag{7}$$

with $\mathbf{u}_k$ denoting the system input, $\mathbf{q}_k$ the system state and $\mathbf{w}_k$ the system noise, which are all in unit quaternion form, namely $\mathbf{q}_k, \mathbf{u}_k, \mathbf{w}_k \in \mathbb{S}^3$. We thus assume the system noise is Bingham-distributed with $\mathbf{w}_k \sim \mathcal{B}(\mathbf{M}_w, \mathbf{Z}_w)$. Here, a general system equation $a(\cdot,\cdot)$ is used, for which the prediction step can be solved by using a deterministic sampling scheme analogous to the unscented transform [15]. However, it would be also interesting to use the technique introduced in II-B for a purely Hamilton product-based system equation without using the sampling scheme, namely

$$\mathbf{q}_k = \mathbf{q}_{k-1} \otimes \mathbf{u}_k \otimes \mathbf{w}_k , \tag{8}$$

where the state $\mathbf{q}_k$ is directly rotated by $\mathbf{u}_k$ and further propagated by the Bingham-distributed system noise $\mathbf{w}_k$ according to the Hamilton product. The corresponding algorithm for the prediction step can thus be proposed as shown in Alg. 1. Here the function for composing two different Bingham distribution (see line 3 in Alg. 1) into a new one is originally introduced in [20].

Algorithm 1 Prediction

procedure predict($\mathcal{B}(\mathbf{M}^e_{k-1}, \mathbf{Z}^e_{k-1}), \mathcal{B}_w, \mathbf{u}_k$)
  $\mathbf{M}_k \leftarrow \mathbf{Q}^r_{u_k}\mathbf{M}^e_{k-1}$;
  $\mathbf{Z}_k \leftarrow \mathbf{Z}^e_{k-1}$;
  $\mathcal{B}(\mathbf{M}^p_k, \mathbf{Z}^p_k) \leftarrow$composeBingham($\mathcal{B}(\mathbf{M}_k, \mathbf{Z}_k), \mathcal{B}_w$);
  return $\mathcal{B}(\mathbf{M}^p_k, \mathbf{Z}^p_k)$
end procedure

B. Progressive Update Step

During the update step, we aim to correct the prior using the measurements. As introduced in [15], the measurement model is restricted to be identity, namely

$$\mathbf{z}_k = \mathbf{q}_k \otimes \mathbf{v}_k \,, \tag{9}$$

with $\mathbf{z}_k, \mathbf{v}_k, \mathbf{q}_k \in \mathbb{S}^3$ and the measurement noise $\mathbf{v}_k$ assumed to be Bingham-distributed . In most of the real-world cases, however, the measurements $\mathbf{z}_k$ are not necessarily in the form of unit quaternions. Besides, the noise term $\mathbf{v}_k$ normally does not follow the previously proposed Bingham distribution after the raw measurements are converted into unit quaternion form. In order to give better modeling of the measurement noise, we introduce an arbitrarily given measurement model

$$\mathbf{z}_k = h(\mathbf{q}_k) \boxplus \mathbf{v}_k \,, \tag{10}$$

with $h(\cdot)$ denoting the measurement function, which maps the system space to the domain of the measurements. Here $\mathbf{v}_k \in \mathbb{Z}$ denotes the measurement noise in an arbitrary domain and $\boxplus$ indicates the composition of both additive and non-additive noise. For instance, a measurement model with the additive noise can be derived based on the Time Differences of Arrival (TDOA) of sensor array for wavefront orientation estimation as shown in (26). The prior can thus be updated according to the Bayes' rule

$$f(\mathbf{q}_k|\mathbf{z}_k) \propto \underbrace{f(\mathbf{z}_k|\mathbf{q}_k)}_{\text{likelihood}} \cdot \underbrace{f(\mathbf{q}_k)}_{\text{prior}} \,, \tag{11}$$

where the likelihood can be derived in a similar way to [18] as

$$f(\mathbf{z}_k|\mathbf{q}_k) = f_{\mathbf{v}_k}\left(h(\mathbf{q}_k)^{-1} \boxplus \mathbf{z}_k\right) \,, \tag{12}$$

with an assumption that the operator $\boxplus$ is invertible. For identity measurement model as (9), its likelihood can then be derived as

$$f(\mathbf{z}_k|\mathbf{q}_k) = f_{\mathbf{v}_k}(\mathbf{q}_k^{-1} \otimes \mathbf{z}_k) \,. \tag{13}$$

This results in the closed-form posterior as a rescaled Bingham distribution as shown in [15]. For non-identity measurement model, instead of the closed-form solution, the update step typically relies on a sampling-approximation-based scheme. However, this method could lead to the problematic sample degeneration and is particularly risky when the sample number is small, e.g., with the Bingham distribution on $\mathbb{S}^3$ only $2 \cdot (4-1)+1 = 7$ samples are drawn deterministically [15].

In order to solve this issue, we can rely on the progressive update methods which have been originally proposed only for Gaussian noise in [21] and further extended to directional state manifolds, e.g., torus [13], circle [22], and manifold of the planar dual quaternions [18]. Detailed introduction can be found in Alg. 2. Instead of updating the prior at once, the filter gradually corrects the prior with the likelihood in multiple progression steps, which is mathematically ensured by the Bayesian inference given (11) as follows

$$f(\mathbf{q}_k|\mathbf{z}_k) = f(\mathbf{q}_k) \cdot \prod_{i=1}^{n} f(\mathbf{z}_k|\mathbf{q}_k)^{\lambda_i} \,, \tag{14}$$

with λ_i indicating each progressive step size and satisfying $\sum_{i=1}^n \lambda_i = 1$, $\lambda_i \in (0,1]$. For each progression step i, we have a predefined threshold τ limiting the ratio of the maximum and minimum rescaled likelihoods $s_j = f(\mathbf{z}_k|\mathbf{q}_j)^{\lambda_i}$ of all the prior samples $\mathbf{q}_j$, namely

$$\frac{\min\limits_{j=1,\cdots,m} f(\mathbf{z}_k|\mathbf{q}_j)^{\lambda_i}}{\max\limits_{j=1,\cdots,m} f(\mathbf{z}_k|\mathbf{q}_j)^{\lambda_i}} = \left(\frac{s_{\text{min}}}{s_{\text{max}}}\right)^{\lambda_i} \geq \tau \,, \tag{15}$$

which gives the step size as

$$\lambda_i \leq \frac{\log(\tau)}{\log(s_{\text{min}}/s_{\text{max}})} \,. \tag{16}$$

We then approximate a new prior using the rescaled likelihoods and redo the sampling-rescaling-approximation scheme for each progression steps until the posterior Bingham is computed in the end.

Algorithm 2 Progressive Update

procedure progressiveUpdate($\mathbf{z}_k, \mathcal{B}(\mathbf{M}^p_k, \mathbf{Z}^p_k), \tau$)
  $\Lambda \leftarrow 1$;
  $i \leftarrow 0$;
  $\mathcal{B}(\mathbf{M}^e_k, \mathbf{Z}^e_k) \leftarrow \mathcal{B}(\mathbf{M}^p_k, \mathbf{Z}^p_k)$;
  while $\Lambda > 0$ **do**
    $i \leftarrow i + 1$;
    $\{(\mathbf{q}_j, p_j)\}_{j=1,\cdots,m} \leftarrow$sampleDeterministic($\mathcal{B}(\mathbf{M}^e_k, \mathbf{Z}^e_k)$);
    $s_{\text{min}} \leftarrow \min\limits_{j=1,\cdots,m} f(\mathbf{z}_k|\mathbf{q}_j)$;
    $s_{\text{max}} \leftarrow \max\limits_{j=1,\cdots,m} f(\mathbf{z}_k|\mathbf{q}_j)$;
    if $s_{\text{max}} = 0$ **then**
      return $\mathcal{B}(\mathbf{M}^e_k, \mathbf{Z}^e_k)$;
    end if
    $\lambda_i \leftarrow \min(\Lambda, \frac{\log(\tau)}{\log(s_{\text{min}}/s_{\text{max}})})$;
    for $j = 1$ **to** m **do**
      $p_j \leftarrow f(\mathbf{z}_k|\mathbf{q}_j)^{\lambda_i} \cdot p_j$;
    end for
    $\mathcal{B}(\mathbf{M}^e_k, \mathbf{Z}^e_k) \leftarrow$ estimateParameters($\{\mathbf{q}_j, p_j\}_{j=1,\cdots,m}$);
    $\Lambda \leftarrow \Lambda - \lambda_i$;
  end while
  return $\mathcal{B}(\mathbf{M}^e_k, \mathbf{Z}^e_k)$;
end procedure

IV. Spherical Average Method on $\mathbb{S}^2$

Instead of using stochastic filtering approaches, it is also intuitive to obtain orientation estimates by averaging noisy orientation measurements. Unlike in the Euclidean space, averaging on the nonlinear manifolds is mathematically not trivial, even for manifolds with relatively simple structure, e.g., the hypersphere. As the unit sphere $\mathbb{S}^2$, where the orientation vector $\mathbf{n}$ locates on, is a compact Riemannian manifold, it would be promising to perform general averaging techniques based on Riemannian geometry through, e.g., geodesic interpolation [23]

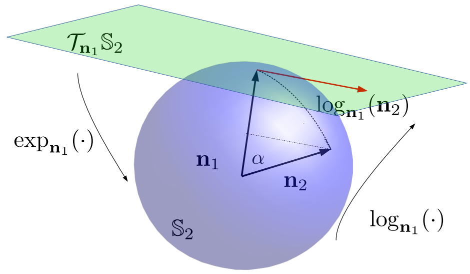

Figure 1: Exponential and logarithm map on unit sphere $\mathbb{S}^2$.

or gradient descent [24]. In this section, we first briefly introduce the necessary mapping techniques such as the exponential and logarithm maps based on spherical geometry. Then we employ the mapping approaches to the standard Riemannian averaging techniques for doing orientation averaging introduced in [16]. This approach has been successfully applied on the $\mathbb{S}^2$ sphere for parameter fitting of the Spherical Normal distribution [16].

A. Logarithm and Exponential Maps on $\mathbb{S}^2$

As shown in Fig. 1 and introduced in [16], for $\forall\, \mathbf{n} \in \mathbb{S}^2$, we can obtain the so-called tangent plane $\mathcal{T}_{\mathbf{n}}\mathbb{S}^2$ that is orthogonal to the unit sphere at $\mathbf{n}$, meaning $\forall\, \mathbf{n}_t \in \mathcal{T}_{\mathbf{n}}\mathbb{S}^2$, $\mathbf{n}_t^\top \mathbf{n} = 0$. $\forall\, \mathbf{n}_1, \mathbf{n}_2 \in \mathbb{S}^2$, the logarithm map that maps $\mathbf{n}_2$ to the tangent plane $\mathcal{T}_{\mathbf{n}_1}\mathbb{S}^2$ determined by $\mathbf{n}_1$ can be derived based on the spherical geometry as

$$\log_{\mathbf{n}_1}(\mathbf{n}_2) = \left(\mathbf{n}_2 - (\mathbf{n}_2^\top \mathbf{n}_1)\mathbf{n}_1\right) \frac{\alpha}{\sin(\alpha)}\,, \tag{17}$$

with $\alpha = \arccos\left(\mathbf{n}_2^\top \mathbf{n}_1\right)$ denoting the angle between the two orientations. Conversely, $\forall\, \mathbf{n}_t \in \mathcal{T}_{\mathbf{n}_1}\mathbb{S}^2$, it can be mapped back to the unit sphere via the exponential map as follows

$$\exp_{\mathbf{n}_1}(\mathbf{n}_t) = \mathbf{n}_1 \cos(||\mathbf{n}_t||) + \frac{\mathbf{n}_t}{||\mathbf{n}_t||} \sin(||\mathbf{n}_t||)\,. \tag{18}$$

It can be verified that both the logarithm and exponential map preserves the geodesic metric, namely the nearest path length between $\forall\, \mathbf{n}_1, \mathbf{n}_2 \in \mathbb{S}^2$ in arc length, i.e.,

$$d(\mathbf{n}_1, \mathbf{n}_2) = \left|\left|\log_{\mathbf{n}_1}(\mathbf{n}_2)\right|\right| = \arccos\left(\mathbf{n}_2^\top \mathbf{n}_1\right)\,. \tag{19}$$

Hereby, the domain of the tangent plane is essentially a circle with a radius of π.

B. Spherical Average Method Based on Gradient Descent

Given a bunch of noisy orientation measurements $\{\mathbf{n}_i\}_{i=1,\dots,N} \in \mathbb{S}^2$, their mean value can be derived by applying general Riemannian gradient descent techniques [25] as shown in Alg. 3 with the aforementioned logarithm and exponential maps. The procedure, which is originally proposed in [16], starts from an arbitrary orientation vector on the sphere (given as the first orientation measurement $\mathbf{n}_1$ in the algorithm), then iteratively minimizes the sum of squared geodesic metric by finding the deepest gradient descent direction on the tangent plane (Alg. 3, line 3), which could be also viewed as a spherical version of the mean shift method.

Algorithm 3 Spherical Average Method

procedure sphericalAverage($\{\mathbf{n}_i\}_{i=1,\dots,N}$)
1: $\bar{\mathbf{n}} \leftarrow \mathbf{n}_1$;
2: **repeat**
3: $\quad \delta\mathbf{n} \leftarrow \frac{1}{N}\sum_{i=1}^{N} \log_{\bar{\mathbf{n}}}(\mathbf{n}_i)$;
4: $\quad \bar{\mathbf{n}} \leftarrow \exp_{\bar{\mathbf{n}}}(\delta\mathbf{n})$;
5: **until** $||\delta\mathbf{n}|| < 10^{-5}$
6: **return** $\bar{\mathbf{n}}$
end procedure

V. Evaluation

In this section, we evaluate the proposed Progressive Bingham Filter (PBF) for spatial wavefront orientation estimation based on simulations. Hereby, the wavefront orientation is measured by a three-dimensional sensor array based on Time Differences of Arrival (TDOA), which can be applied to a variety of practical scenarios, e.g., multilateration, sensor network localization, etc. We then compare the proposed PBF to the Spherical Average Method (SAM) as well as the ordinary Bingham Filter (BF), which assumes the identity measurement model.

A. Wavefront Orientation Measurement Based on Time Differences of Arrival (TDOA)

When the source is located sufficiently far away from the sensor array compared to the array size, we can assume that the wavefront can be approximated as a plane in the three-dimensional Euclidean space, which is determined by

$$\mathbf{n}^\top \mathbf{s}_i + b = r_i\,, \tag{20}$$

where $i = 1, \cdots, S$ for S sensors, $\mathbf{n}$ denotes the unit vector indicating the wavefront orientation, the scalar b denotes the plane offset and r_i is the signed distance of sensor location $\mathbf{s}_i$ to the plane. For an arbitrary sensor pair $\forall\, i, j \in 1, \cdots, S$ of the sensor network, their Time Differences of Arrival (TDOA) can be derived as

$$\begin{aligned} t_{i,j} = t_i - t_j &= \frac{1}{c}(\mathbf{s}_i - \mathbf{s}_j)^\top \mathbf{n} \\ &= \frac{1}{c}\mathbf{s}_{i,j}^\top \mathbf{n}\,, \end{aligned} \tag{21}$$

with c denoting the propagation speed and $\mathbf{s}_{i,j}$ the vector from sensor i to sensor j. We can thus derive the measurement model for the wavefront orientation by concatenating the TDOA equation of each possible sensor pair

$$\begin{bmatrix} \mathbf{t}_{1,\{2,\cdots,S\}} \\ \mathbf{t}_{2,\{3,\cdots,S\}} \\ \vdots \\ t_{S-1,S} \end{bmatrix}_{N\times 1} = \frac{1}{c} \begin{bmatrix} \mathbf{s}_{1,\{2,\cdots,S\}}^\top \\ \mathbf{s}_{2,\{3,\cdots,S\}}^\top \\ \vdots \\ \mathbf{s}_{S-1,S}^\top \end{bmatrix}_{N\times 3} \cdot \mathbf{n}\,, \tag{22}$$

with $\mathbf{n} \in \mathbb{S}^2$ and $N = \binom{S}{2}$. Here, e.g., $\mathbf{t}_{1,\{2,\cdots,S\}}$ denotes the TDOA measurements between the first sensor and the rest of the sensors. We then formulate the aforementioned equation for sensor cluster $\{\mathbf{s}_i\}_{i=1,\cdots N}$ in a concise manner as

$$\Delta\mathbf{t} = \mathbf{H} \cdot \mathbf{n}\,, \tag{23}$$

where $\Delta\mathbf{t}$ denotes the TDOA for each pair of the sensors in the array, $\mathbf{H}$ is determined by the sensor array structure which is constant, and $\mathbf{n} \in \mathbb{S}^2$ denotes the resulting wavefront orientation unit vector. Given the TDOA measurements, the wavefront orientation can be directly calculated according to the following closed-form solution

$$\bar{\mathbf{n}} = \mathbf{H}^{+}\Delta\mathbf{t}\,, \tag{24}$$

with $\mathbf{H}^{+}$ denoting the pseudoinverse of $\mathbf{H}$, namely $\mathbf{H} = (\mathbf{H}^{\top}\mathbf{H})^{-1}\mathbf{H}^{\top}$. Note for noisy measurement, the resulting $\bar{\mathbf{n}}$ does not fall on the unit sphere anymore, thus a post normalization is necessary.

B. Evaluation Set-up

In order to evaluate the recursive estimators based on the Bingham distribution and compare them to non-stochastic approaches, we use unit quaternions to represent the wavefront orientations as the system state. The wavefront is simulated to follow the Bingham distribution, namely

$$\mathbf{q}_{k+1} = \mathbf{q}_k \otimes \mathbf{w}_k, \quad \mathbf{w}_k \sim \mathcal{B}(\mathbf{M}_w, \mathbf{Z}_w)\,, \tag{25}$$

with $\mathbf{M}_w\mathbf{Z}_w\mathbf{M}_w^{\top} = \mathrm{diag}(0, -100, -100, -100)$. The individual parameter matrix, $\mathbf{M}_w$ and $\mathbf{Z}_w$, can be calculated through eigendecomposition of the diagonal matrix. This distribution is thus zero-centered, i.e., has the mode at $[1, 0, 0, 0]^{\top}$. Based on the TDOA function formulated in (23), we set the measurement model to

$$\mathbf{z}_k = \mathbf{H} \cdot (\mathbf{q}_k \otimes \mathbf{n}_0 \otimes \mathbf{q}_k^{*}) + \mathbf{v}_k, \quad \mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \Sigma_v)\,, \tag{26}$$

with $\mathbf{n}_0 = [1, 0, 0]^{\top}$ denoting the initialized orientation such that the wavefront orientation can be represented by the unit quaternion $\mathbf{q}_k$ via $\mathbf{n} = \mathbf{q}_k \otimes \mathbf{n}_0 \otimes \mathbf{q}_k^{*}$. The sensor array has a spherical structure as shown exemplarily in Fig. 2, where the six individual sensors are homogeneously located on the surface. The measurements $\mathbf{z}_k \in \mathbb{R}^N$ ($N = \binom{6}{2} = 12$) are essentially the TDOAs from the sensor array. Each individual TDOA measurement is assumed to be independently Gaussian-distributed and has zero mean and variance $\sigma^2 = 0.5$ (thus Σ_v in (26) is diagonal).

The proposed Progressive Bingham Filter (PBF) is then compared with the original Bingham Filter (BF), where the identity measurement model is assumed, i.e. $\mathbf{z}_k = \mathbf{q}_k \otimes \mathbf{v}_k$. In order to have similar noise level, we approximate the Bingham distribution $\mathcal{B}_v$ of identity measurement model based on Monte Carlo runs of the measurement model in (25). To get the measurements for the BF, we first calculate the rotation angle relative to the initialized orientation through $\theta = \mathrm{acos}(\mathbf{n}_0^{\top}\mathbf{n})$ and the rotation axis through $\mathbf{u} = \mathbf{n}_0 \times \mathbf{n}$, then compose the pseudo measurements in unit quaternion form according to the definition in (1), such that $\mathbf{n} = \mathbf{q} \otimes \mathbf{n}_0 \otimes \mathbf{q}^{*}$. Besides the stochastic approaches, we also evaluate the proposed Spherical Averaging Method (SAM) where multiple orientation measurements are collected from several sensors of the array based on (23) and then get averaged on $\mathbb{S}^2$. As a baseline, we also calculate the result according to the closed-form solution in (24).

C. Evaluation Result

Fig. 3 shows the evaluation results based on 200 Monte Carlo runs. We give the geodesic RMSE based on the metric

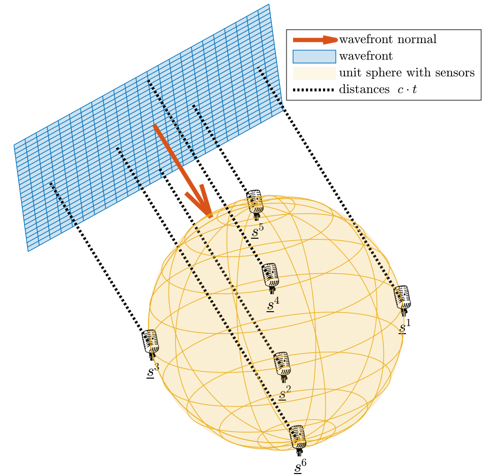

Figure 2: An example of the wavefront orientation measured by the sensor array structured on the sphere. The sensors can also be arranged according to other array structures.

in (19). The proposed progressive filter outperforms the original one due to the fact that the PBF directly models the measurement noise whereas the BF still assumes the measurement model to be identity. The Spherical Averaging Method (SAM) gives rational result, however, worse than the stochastic approaches. This is not surprising, because the recursive estimators consider the system propagation with the assumed Bingham-distributed noise in (25).

VI. Conclusion and Outlook

In this work, we proposed a novel stochastic orientation estimator, the Progressive Bingham Filter (PBF), which is based on the unit quaternion representation. Compared to the previously proposed Bingham Filter (BF) [15] where the measurement model is assumed to be the identity, it can handle arbitrary measurement models which allow direct modeling of the measurement noise. This is done by applying the progressive update approach based on deterministic sampling scheme on $\mathbb{S}^3$, which is able to resolve the sample degeneration issue. Furthermore, we point out that the sampling-approximation-based propagation scheme can be simplified to be closed-form if the system equation is purely Hamilton product-based. Besides the stochastic approaches, we also employ an averaging technique for spatial orientation, namely the Spherical Averaging Method (SAM) derived based on Riemannian geometry. As an evaluation, we applied the proposed methods for orientation estimation given TDOA measurements, which could be of great interest to the surveillance field. The simulation results show better performance of the proposed Progressive Bingham Filter. This indicate the possibility, at least theoretically, of applying directional statistics-based method to solve wavefront orientation estimation problems using the TDOA measurements.

There is still much potential that can be exploited based on this work. For instance, the SAM approach can be extended from $\mathbb{S}^2$, where the orientation vectors are located on, to the manifold of unit quaternion $\mathbb{S}^3$ and further integrated into the stochastic filtering scheme for general averaging purposes. The proposed Progressive Bingham Filter should be further tested in real-world scenario for TDOA-based orientation finding application. Moreover, the proposed orientation estimator can be extended to perform localization in a time-of-arrival-based (TOA-based) sensor network. Once the wavefront orientation can be estimated by each sensor node individually, the source location can thus be determined accordingly with proper distributed estimation methods.

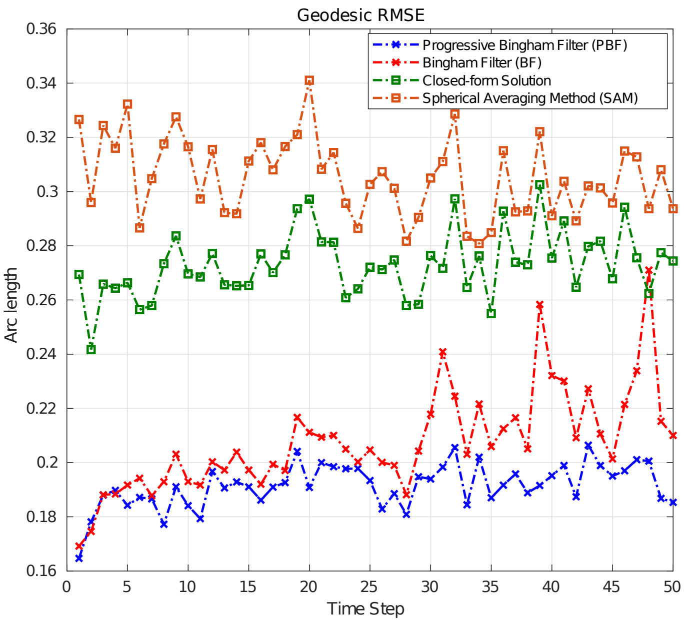

Figure 3: Evaluation results based on 200 Monte Carlo runs. As baseline, the green curve is given by the closed-form solution introduced in Sec. V-A. The orange curve is from the SAM introduced in Sec. IV. The red curve is given by the Bingham Filter with identity measurement model [15]. The proposed progressive approach (blue curve) outperforms the other methods.

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