New Results for State Estimation in the Presence of Mixed Stochastic and Set Theoretic Uncertainties

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ABSTRACT

This paper presents a new approach for estimating the state of a linear dynamic system when two different types of uncertainties are present simultaneously. The first type of uncertainty is a stochastic process with given distribution. The second type of uncertainty is only known to be bounded, the exact underlying distribution is unknown. This includes inequality constraints between state variables, geometric tolerances, and bounded noise sources which are possibly correlated. For this generalized uncertainty model, a new recursive estimator has been developed comprising time and measurement update. The new estimator unifies Kalman filtering and set theoretic filtering. It converges to a Kalman filter, when the bounded uncertainty goes to zero, and it converges to a set theoretic filter, when the stochastic noise vanishes. In the case of mixed uncertainties, the new estimator provides solution sets that are uncertain in a statistical sense.

1 INTRODUCTION

We consider the problem of determining the state of a discrete-time system according to

$$\underline{x}_{k+1} = \mathbf{A}_k \underline{x}_k + \mathbf{B}_k \underline{u}_k$$

with scalar measurement equation

$$y_k = \underline{H}_k^T \underline{x}_k \ ,$$

where $\underline{x}_k$ denotes the state vector at time step k, $\underline{u}_k$ denotes the system input at time step k, and y_k denotes the observation at time step k. When the system is fully observable, the state $\underline{x}_k$ can be reconstructed in the noise-free case based on a sufficient number of observations up to time step k using for example a Luenberger observer [13].

In general, however, system states and observations are corrupted by uncertainties according to

$$\begin{aligned} \underline{x}_{k+1} &= \mathbf{A}_k \underline{x}_k + \mathbf{B}_k \left\{ \underline{u}_k + \underline{w}_k \right\} \ , \\ y_k &= \underline{H}_k^T \underline{x}_k + v_k \ , \end{aligned} \tag{1}$$

where $\underline{w}_k$ denotes the *additive* input uncertainties and v_k denotes the *additive* output uncertainties.

These uncertainties can, for example, be described in a stochastic setting. A stochastic model is, for example, appropriate for describing thermal noise. In that case, a Kalman filter can be used for estimating the system state [12].

On the other hand, the uncertainties can be modeled as being bounded with no underlying distribution assumed [1]. This is useful for including inequality constraints between state variables, geometric tolerances, and bounded noise sources which are possibly correlated. For the case of bounded uncertainties, a set theoretic filter is the appropriate tool for estimating the system state [15].

Many real-world problems can be described by a combination of the two types of uncertainties, i.e., of stochastic and set theoretic uncertainties. This is essential when including noisy constraints or when considering the additive combination of noise with known distribution and noise with known bounds.

In [5, 8], a concept for state estimation in the presence of both set theoretic and stochastic uncertainties has been introduced. The proposed algorithm for the case of a scalar state is exact, but computationally complex. In [6, 7], an approximate solution for the case of a scalar state has been derived, that is computationally attractive. Furthermore, a generalization towards arbitrary dimensional states and observations of the same dimension has been proposed in [9].

This paper is concerned with arbitrary dimensional states $\underline{x}_k$ and scalar observations y_k. For this very relevant case, a new, approximate solution is derived, that is computationally attractive. Nevertheless, it combines both stochastic and set theoretic estimation in a rigorous manner. It bridges the gap between both estimation schemes, because a Kalman filter is attained,

[1]This is different from assuming uniformly distributed random variables!

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when the bounded error goes to zero, and a set theoretic estimator is attained, when the stochastic error vanishes. When both types of uncertainty are present, the new estimator provides solution sets that are uncertain in a statistical sense.

In Section 2, the Kalman filter for recursively estimating the system state in the presence of stochastic uncertainties is reviewed for the case of a linear system and scalar observations. Section 3 presents a review of the set theoretic filter for state estimation with bounded uncertainties also for the case of a linear system and scalar observations. The new filter is then introduced in Section 4. In Section 5, a two-dimensional simulative example is presented that further clarifies the conveyed concepts.

2 KALMAN FILTER

When a stochastic noise model is adopted, a Kalman filter is appropriate for estimating the system state. Here, we have

$$\underline{w}_k = \underline{c}_k^u \ ,$$
$$v_k = c_k^y \ ,$$

where $\underline{c}_k^u$, c_k^y are assumed to be

1) zero mean
2) independent
3) Gaussian distributed

random variables with known covariances according to

$$\underline{c}_k^u \sim \underline{N}(\underline{0},\ \mathbf{C}_k^u), \quad c_k^y \sim N(0,\ C_k^y) \ .$$

Of course, it is possible to drop the independence assumption and just call for known correlation between $\underline{c}_k^u$ and c_k^y. Furthermore, it is also possible to drop the Gaussian assumption and assume given moments up to second order.

2.1 Time Update

Given a state estimate $\underline{x}_{k-1}^f$ at time $k-1$, the system model (1) is used to perform the so-called time update

$$\underline{x}_k^p = \mathbf{A}_{k-1}\underline{x}_{k-1}^f + \mathbf{B}_{k-1}\underline{u}_{k-1} \ ,$$

with covariance matrix

$$\mathbf{C}_k^p = \mathbf{A}_{k-1}\mathbf{C}_{k-1}^f\mathbf{A}_{k-1}^T + \mathbf{B}_{k-1}\mathbf{C}_{k-1}^u\mathbf{B}_{k-1}^T \ .$$

The time update is started with an initial noisy state described by $\underline{x}_0^f$, $\mathbf{C}_0^f$.

2.2 Measurement Update

Subsequently, the observation y_k at time k is used to perform the measurement update according to

$$\underline{x}_k^f = \underline{x}_k^p + \frac{\mathbf{C}_k^p\underline{H}_k}{C_k^y + \underline{H}_k^T\mathbf{C}_k^p\underline{H}_k}(y_k - \underline{H}_k^T\underline{x}_k^p) \ ,$$

with the following recursion for the covariance matrix

$$\mathbf{C}_k^f = \mathbf{C}_k^p - \frac{\mathbf{C}_k^p\underline{H}_k\,\underline{H}_k^T\mathbf{C}_k^p}{C_k^y + \underline{H}_k^T\mathbf{C}_k^p\underline{H}_k} \ .$$

3 SET THEORETIC FILTER

In the case of a bounded uncertainty model, a set theoretic filter is appropriate for estimating the system state. Here, we have

$$\underline{w}_k = \underline{e}_k^u \ ,$$
$$v_k = e_k^y \ ,$$

where we assume no prior information about $\underline{e}_k^u$, e_k^y besides that they are bounded according to

$$\left(\underline{e}_k^u\right)^T\left(\mathbf{E}_k^u\right)^{-1}\underline{e}_k^u \leq 1 \ , \quad \left(e_k^y\right)^2 \leq E_k^y \ .$$

Essentially, this means that the underlying distributions of the individual variables $\underline{e}_k^u$, e_k^y and their joint distribution are completely unknown. Hence, this class of uncertainties includes systematic, correlated, and fully dependent errors.

3.1 Time Update

Given a state estimate $\underline{x}_{k-1}^f$ at time $k-1$, the system model (1) is used to perform the time update

$$\underline{x}_k^p = \mathbf{A}_{k-1}\underline{x}_{k-1}^f + \mathbf{B}_{k-1}\underline{u}_{k-1} \ ,$$

with

$$\mathbf{E}_k^p = \frac{1}{0.5-\kappa_k}\mathbf{A}_{k-1}\mathbf{E}_{k-1}^f\mathbf{A}_{k-1}^T + \frac{1}{0.5+\kappa_k}\mathbf{B}_{k-1}\mathbf{E}_{k-1}^u\mathbf{B}_{k-1}^T \ .$$

$\kappa_k \in (-0.5, 0.5)$ is selected in such a way, that the size of $\mathbf{E}_k^p$ is minimized [5]. The time update is started with an initial set of states described by $\underline{x}_0^f$, $\mathbf{E}_0^f$.

3.2 Measurement Update

Subsequently, the observation y_k at time k is used to perform the measurement update according to

$$\underline{x}_k^f = \underline{x}_k^p + \lambda_k\frac{\mathbf{E}_k^p\underline{H}_k}{E_k^y + \lambda_k\,\underline{H}_k^T\mathbf{E}_k^p\underline{H}_k}\left(y_k - \underline{H}_k^T\underline{x}_k^p\right)$$

and

$$\mathbf{E}_k^f = d_k\mathbf{P}_k^f$$

with

$$\mathbf{P}_k^f = \mathbf{E}_k^p - \lambda_k\frac{\mathbf{E}_k^p\underline{H}_k\,\underline{H}_k^T\mathbf{E}_k^p}{E_k^y + \lambda_k\,\underline{H}_k^T\mathbf{E}_k^p\underline{H}_k} \ .$$

and

$$d_k = 1 + \lambda_k - \frac{\lambda_k\left(y_k - \underline{H}_k^T\underline{x}_k^p\right)^2}{E_k^y + \lambda_k\,\underline{H}_k^T\mathbf{E}_k^p\underline{H}_k}$$

$\lambda_k \in [0, \infty)$ is chosen to minimize the size of $\mathbf{E}_k^f$ [5].

4 THE NEW FILTER

Now, we consider a combined uncertainty model [2, 3] with

$$\underline{w}_k = \underline{e}^u_k + \underline{c}^u_k \ , \qquad v_k = e^y_k + c^y_k \ ,$$

where $\underline{e}^u_k$, e^y_k are bounded according to

$$\left(\underline{e}^u_k\right)^T \left(\mathbf{E}^u_k\right)^{-1} \underline{e}^u_k \le 1 \ , \quad \left(e^y_k\right)^2 \le E^y_k \ ,$$

and where $\underline{c}^u_k$, c^y_k are assumed to be

1) zero mean
2) independent
3) Gaussian distributed

random variables with known covariances according to

$$\underline{c}^u_k \sim \underline{N}(\underline{0},\ \mathbf{C}^u_k), \quad c^y_k \sim N(0,\ C^y_k) \ .$$

4.1 Time Update

The time update for a combined uncertainty model can be performed by superposition. Hence, the result is simply given by

$$\underline{x}^p_k = \mathbf{A}_{k-1}\underline{x}^f_{k-1} + \mathbf{B}_{k-1}\underline{u}_{k-1} \ .$$

For the covariance of the predicted state $\underline{x}^p_k$ we have

$$\mathbf{C}^p_k = \mathbf{A}_{k-1}\mathbf{C}^f_{k-1}\mathbf{A}^T_{k-1} + \mathbf{B}_{k-1}\mathbf{C}^u_{k-1}\mathbf{B}^T_{k-1} \ .$$

Its set theoretic uncertainty is calculated according to

$$\mathbf{E}^p_k = \frac{1}{0.5-\kappa_k}\mathbf{A}_{k-1}\mathbf{E}^f_{k-1}\mathbf{A}^T_{k-1} + \frac{1}{0.5+\kappa_k}\mathbf{B}_{k-1}\mathbf{E}^u_{k-1}\mathbf{B}^T_{k-1} \ .$$

Again, $\kappa_k \in (-0.5, 0.5)$ is selected in such a way, that the size of $\mathbf{E}^p_k$ is minimized. The time update is started with an initial noisy set of states described by $\underline{x}^f_0$, $\mathbf{C}^f_0$, $\mathbf{E}^f_0$.

4.2 Measurement Update

In contrast to the time update, the generalization of the measurement update is *not* simply a combination of the update formulae of Kalman filter and set theoretic filter. The update step conceptually is performed by intersecting two sets with random position. Of course, the update result is a complicated set with random size, orientation, and position. This exact result is approximated to second order, i.e., by an ellipsoidal set with a Gaussian distributed random midpoint. The mean of the midpoint is given by

$$\underline{x}^f_k = \mathbf{W}^p_k\underline{x}^p_k + \underline{W}^y_k y - \left(\mathbf{W}^p_k\mathbf{C}^p_k\underline{H} - \underline{W}^y_k C^y_k\right) F_1\left(y_k - \underline{H}^T_k\underline{x}^p_k\right) \qquad (2)$$

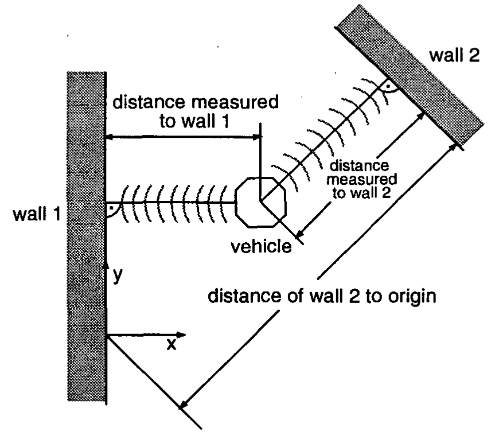

Figure 1: Setup for simulative example.

with an associated covariance matrix

$$\mathbf{C}^f_k = \mathbf{W}^p_k\mathbf{C}^p_k\left(\mathbf{W}^p_k\right)^T + \underline{W}^y_k\left(\underline{W}^y_k\right)^T C^y_k - \left(\mathbf{W}^p_k\mathbf{C}^p_k\underline{H}_k - \underline{W}^y_k C^y_k\right) \left(\mathbf{W}^p_k\mathbf{C}^p_k\underline{H}_k - \underline{W}^y_k C^y_k\right)^T F_2\left(y_k - \underline{H}^T_k\underline{x}^p_k\right) \ , \qquad (3)$$

where $\mathbf{W}^p_k$, $\underline{W}^y_k$, $F_1\left(y_k - \underline{H}^T_k\underline{x}^p_k\right)$, $F_2\left(y_k - \underline{H}^T_k\underline{x}^p_k\right)$ are given in the appendix. For the set theoretic uncertainty we have

$$\mathbf{E}^f_k = (1+\lambda_k)\,\mathbf{E}^p_k - (1+\lambda_k)\,\lambda_k\,\frac{\mathbf{E}^p_k\underline{H}_k\,\underline{H}^T_k\mathbf{E}^p_k}{E^y_k + \lambda_k\,\underline{H}^T_k\mathbf{E}^p_k\underline{H}_k} \ , \qquad (4)$$

which defines the size and orientation of the ellipsoidal set. $\lambda_k \in [0, \infty)$ is chosen to minimize a certain function of $\mathbf{C}^f_k$ and $\mathbf{E}^f_k$.

5 SIMULATIVE EXAMPLE

Consider a vehicle equipped with range sensors that measure the distances to two walls i, $i = 1, 2$, Figure 1. The wall positions are known within a given geometric tolerance, i.e.,

$$d_i = \tilde{d}_i + \Delta d_i \ , \text{ with } |\Delta d_i| \le b_i \ ,$$

where $\tilde{d}_i$ denotes the unknown true (signed) distance of the wall to the origin and Δd_i is the unknown but bounded deviation of the nominal value d_i. The corresponding unit normal vector $\underline{H}_i$ is assumed to be known. The range measurements are corrupted by additive white Gaussian noise with zero mean and a variance σ_i^2 which depends on the surface characteristics of wall i. The measurement equation is thus given by

$$d_i + D^k_i = \underline{H}^T_i\tilde{\underline{x}} + \Delta d_i + c^k_i \ ,$$

where $c^k_i \sim N(0, \sigma_i)$, $\tilde{\underline{x}}$ denotes the vehicle position, and D_i is the measured distance. A true vehicle position $\tilde{\underline{x}} = [2000, 2000]^T$ is assumed. The remaining parameters are given in Tab. 1.

Wall	1	2
Unit normal vector $\underline{H}_i$	$[1,0]^T$	$-1/\sqrt{2}[1,1]^T$
Nominal distance d_i	0	-6000
True distance $\tilde{d}_i$	-40	-6030
Bound b_i	50	50
Standard deviation σ_i	100	100

Table 1: Parameters of localization experiment.

The initial predicted position is given by $\underline{x}_0^p = [1900, 2100]^T$ with $\mathbf{E}_0^p = \text{diag}\left(2000^2, 2000^2\right)$ and $\mathbf{C}_0^p = \text{diag}\left(2000^2,\ 2000^2\right)$. At each time instant k, the distances to both walls are measured.

The proposed new estimator is evaluated by recursively updating the position estimate using the equation for $\underline{x}_k^f$ in (2), $\mathbf{E}_k^f$ in (4), and $\mathbf{C}_k^f$ in (4) twice: Once for wall 1 with $E_k^y = b_1^2$ and $C_k^y = \sigma_1^2$, which yields an intermediate estimate, and once for wall 2 with $E_k^y = b_2^2$ and $C_k^y = \sigma_2^2$, which yields the estimate $\underline{x}_k^f$ that incorporates all measurements available up to time k. The parameter λ_k is chosen such that $|\mathbf{E}_k^f| + |\mathbf{C}_k^f|$ is minimized. Figure 2 depicts how the resulting estimate evolves over time. Please note, that the state is kept constant for this problem to make interpretation easy. Hence, no time update is necessary.

The confidence set, i.e., the Minkowski sum of E_k^f and $9\,C_k^f$ centered at $\underline{x}_k^f$, is given for $k = 1, 2, 3, 10, 1000$. The optimal estimate for an infinite number of measurements would be the set resulting from intersecting the two strips that correspond to the uncertainty of the two walls. The exact state $\tilde{\underline{x}} = [2000, 2000]^T$ is marked by a dot. **Note:** The confidence set for $k \to \infty$ bounds the exact set from above and hence contains the true state.

To compare these results with standard Kalman filtering, we view the wall uncertainty as an additional uncorrelated noise term with zero mean and variance E_k^y, which results in a total measurement variance of $C_k^y + E_k^y$. The evolution of the resulting confidence set is depicted in Figure 3. Here, the confidence set has been calculated based on 9 times the Kalman filter covariance matrix centered at $\underline{x}_k^f$. **Note:** The confidence set for $k \to \infty$ does *not* contain the true state.

Due to limited space, the result of applying purely set theoretic filtering to this problem is not presented. Of course, the result is too pessimistic, since independence of the range measurements is not exploited.

6 CONCLUSIONS

A vast class of estimation problems can be attacked as a mixed noise problem, i.e., the arising uncertainties can be modeled as being additively composed of both 1) noise with known distribution and 2) noise with known bounds. For these problems, a new estimator

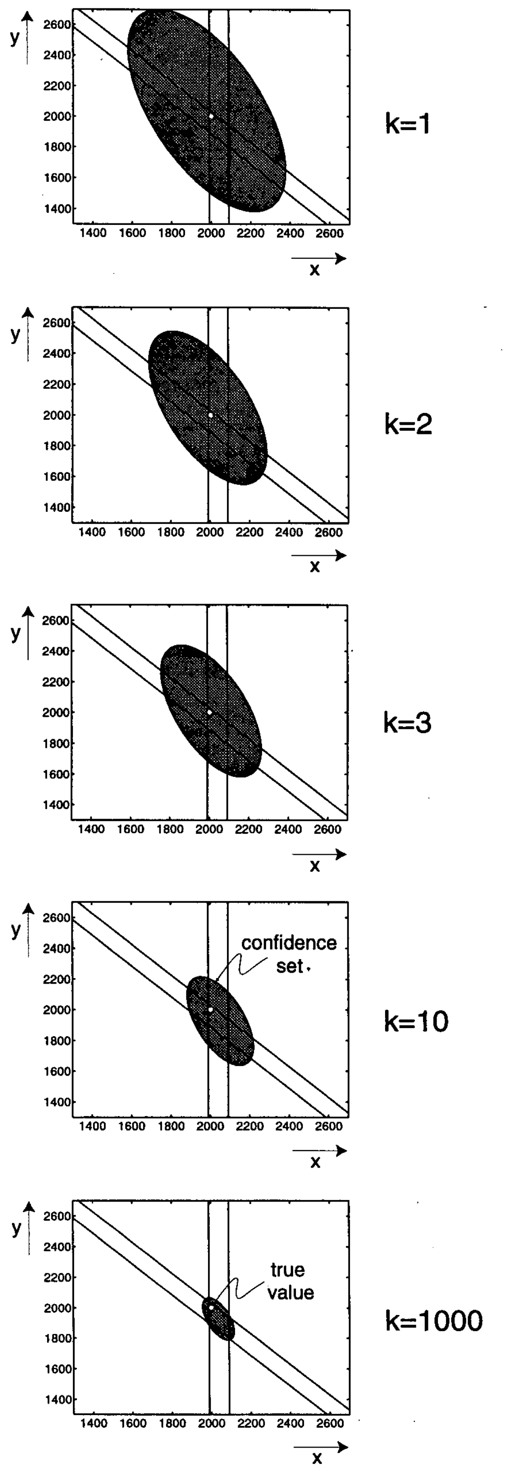

Figure 2: Results of applying the new estimator: Evolution of confidence sets over time.

has been derived for the important case of linear systems with arbitrary dimensional states and scalar measurements. The estimator provides solution sets with Gaussian distributed random positions. Of course, the new estimator contains the Kalman filter and the set theoretic filter as border cases.

The new estimator provides a computationally attractive means for solving mixed uncertainty problems in a rigorous manner.

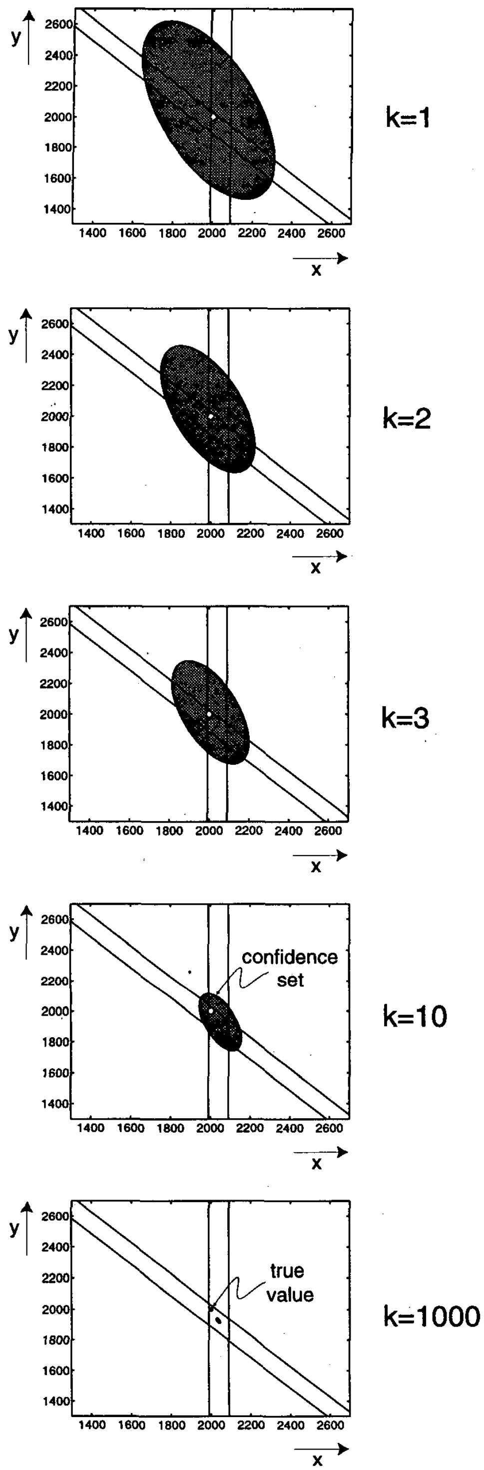

Figure 3: Results of Kalman filtering: Evolution of confidence sets over time.

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7 APPENDIX

For the weighting factors $\mathbf{W}_k^p$, $\underline{W}_k^y$ we have

$$\mathbf{W}_k^p = I - \lambda_k \frac{\mathbf{E}_k^p \underline{H}_k \underline{H}_k^T}{E_y + \lambda_k \underline{H}^T \mathbf{E}_p \underline{H}} \,,$$

$$\underline{W}_k^y = \frac{\lambda_k \mathbf{E}_k^p \underline{H}_k}{E_k^y + \lambda_k \underline{H}_k^T \mathbf{E}_k^p \underline{H}_k} \,.$$

The nonlinear functions $F_1\left(y_k - \underline{H}_k^T \underline{x}_k^p\right)$, $F_2\left(y_k - \underline{H}_k^T \underline{x}_k^p\right)$ of the innovation $y_k - \underline{H}_k^T \underline{x}_k^p$ are given by

$$F_1\left(y_k - \underline{H}_k^T \underline{x}_k^p\right) = G_0\left(y_k - \underline{H}_k^T \underline{x}_k^p, \sqrt{E_k^y} + \sqrt{\underline{H}_k^T \mathbf{E}_k^p \underline{H}_k}, \sqrt{\underline{H}_k^T \mathbf{C}_k^p \underline{H}_k + C_k^y}\right) \,,$$

$$F_2\left(y_k - \underline{H}_k^T \underline{x}_k^p\right) = \left[G_0\left(y_k - \underline{H}_k^T \underline{x}_k^p, \sqrt{E_k^y} + \sqrt{\underline{H}_k^T \mathbf{E}_k^p \underline{H}_k}, \sqrt{\underline{H}_k^T \mathbf{C}_k^p \underline{H}_k + C_k^y}\right)\right]^2 + \frac{G_1\left(y_k - \underline{H}_k^T \underline{x}_k^p, \sqrt{E_k^y} + \sqrt{\underline{H}_k^T \mathbf{E}_k^p \underline{H}_k}, \sqrt{\underline{H}_k^T \mathbf{C}_k^p \underline{H}_k + C_k^y}\right)}{\underline{H}_k^T \mathbf{C}_k^p \underline{H}_k + C_k^y} \,,$$

with functions G_0 and G_1

$$G_0(x, B, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \frac{\exp\left\{-\frac{1}{2}\frac{(x-B)^2}{\sigma^2}\right\} - \exp\left\{-\frac{1}{2}\frac{(x+B)^2}{\sigma^2}\right\}}{\operatorname{erf}\left\{\frac{x-B}{\sigma}\right\} - \operatorname{erf}\left\{\frac{x+B}{\sigma}\right\}} \,,$$

$$G_1(x, B, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \frac{(x-B)\exp\left\{-\frac{1}{2}\frac{(x-B)^2}{\sigma^2}\right\} - (x+B)\exp\left\{-\frac{1}{2}\frac{(x+B)^2}{\sigma^2}\right\}}{\operatorname{erf}\left\{\frac{x-B}{\sigma}\right\} - \operatorname{erf}\left\{\frac{x+B}{\sigma}\right\}} \,.$$