A New State Estimator for a Mixed Stochastic and Set Theoretic Uncertainty Model

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ABSTRACT

This work presents new results for state estimation based on noisy observations suffering from two different types of uncertainties. The first uncertainty is a stochastic process with given statistics. The second uncertainty is only known to be bounded, the exact underlying statistics are unknown. State estimation tasks of this kind typically arise in target localization, navigation, and sensor data fusion. A new estimator has been developed, that combines set theoretic and stochastic estimation in a rigorous manner. The estimator is efficient and, hence, well-suited for practical applications. It provides a continuous transition between the two classical estimation concepts, because it converges to a set theoretic estimator, when the stochastic error goes to zero, and to a Kalman filter, when the bounded error vanishes. In the mixed noise case, the new estimator provides solution sets that are uncertain in a statistical sense.

Keywords: State estimation, mixed uncertainty model, sensor fusion, vehicle localization

1. INTRODUCTION

In many applications, it is important to deduce a system's state on the basis of uncertain observations of the system's output. In addition, uncertain results of different estimators must be combined. Applications include vehicle or missile localization, target tracking, navigation, and sensor data fusion. The goal of these estimation procedures is, of course, to reduce the uncertainty about the system's state as much as possible.

When an appropriate system model together with noise statistics is given, the Kalman filter and its descendants[1] have been successfully applied for more than 30 years. However, in the applications cited above, a detailed statistical noise model is often either not available or impractical. Special caution is in order when neglecting strongly correlated noise or systematic errors. In that case, Kalman filter estimates tend to be overoptimistic,[6] i.e., the covariance estimate becomes unrealistically small. Several heuristics have been suggested for coping with this problem, ranging from artificially increasing the covariance from time to time to employing nonlinear pre–filters. Of course, these techniques do not provide optimal estimators.

In some situations, although a statistical noise description cannot be given, bounds for the noise can be provided. This may be the case for unmodeled dynamics, unmodeled nonlinearities, correlated noise, and systematic errors. In that case, set theoretic estimation can be applied,[8] which often leads to good results.[3] However, when additional uncorrelated noise is present, the error bounds become unnecessarily conservative.

This paper introduces a new estimator that combines both set theoretic and stochastic estimation in a rigorous manner. It bridges the gap between both estimation schemes, because it becomes a set theoretic estimator, when the stochastic error goes to zero, and a Kalman filter, when the bounded error vanishes. When both types of uncertainty are present, the new estimator provides solution sets that are uncertain in a statistical sense.

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2. DERIVATION OF THE NEW ESTIMATOR

We restrict attention to the scalar case for the sake of simplicity. Without loss of generality we consider two uncertain measurements of an unknown state z given by

$$\begin{aligned} x &= z + e_x + c_x,\, e_x^2 \le E_x,\, c_x \sim N(0, \sigma_x) \ , \\ y &= z + e_y + c_y,\, e_y^2 \le E_y,\, c_y \sim N(0, \sigma_y) \ . \end{aligned} \tag{1}$$

x and y suffer from two types of additive noise[2]: 1) Uncertainties e_x, e_y, where the only prior knowledge is their boundedness and 2) Gaussian noise sources c_x, c_y, which are assumed to be uncorrelated.

First, assume that x, y can be observed without stochastic uncertainty. Then, since there is no prior information about e_x, e_y besides their boundedness, we make the worst case assumption that e_x, e_y are fully correlated. In that case, a set theoretic estimator is appropriate for fusing the information sources. An efficient form of a set theoretic estimator, which is based on the convex combination of the original sets, is given by the set[8]

$$\mathcal{Z} = \{z : (z - \hat{z})^2 \le E_z\} \ , \tag{2}$$

which is an interval in the scalar case. The interval midpoint is given by

$$\hat{z} = w_x x + w_y y \tag{3}$$

with weighting factors

$$\begin{aligned} w_x &= (0.5 - \lambda)\frac{P_z}{E_x} \ , \\ w_y &= (0.5 + \lambda)\frac{P_z}{E_y} \end{aligned} \tag{4}$$

with

$$P_z = \left(\frac{0.5 - \lambda}{E_x} + \frac{0.5 + \lambda}{E_y} \right)^{-1} \ , \tag{5}$$

where $w_x + w_y = 1$. The appropriate selection of the parameter $\lambda \in [-0.5, 0.5]$ will be discussed later. The set theoretic uncertainty is given by

$$E_z = d\, P_z \tag{6}$$

with

$$d = 1 - \left(0.25 - \lambda^2\right) (x - y)^2 \frac{P_z}{E_x E_y} \ . \tag{7}$$

It is important to note that the set theoretic uncertainty E_z depends on the actual observations x, y. From now on, we will bound E_z from above by setting $d = 1$. The so obtained interval always contains the true interval. Most importantly, E_z does not depend on the actual observations, which simplifies some of the following derivations. Hence, we obtain the interval

$$\mathcal{Z} = \{z : (z - \hat{z})^2 \le E_z\} \ , \tag{8}$$

with midpoint

$$\hat{z} = w_x x + w_y y \tag{9}$$

and weighting factors

$$\begin{aligned} w_x &= (0.5 - \lambda)\frac{E_z}{E_x} \ , \\ w_y &= (0.5 + \lambda)\frac{E_z}{E_y} \ , \end{aligned} \tag{10}$$

for $\lambda \in [-0.5, 0.5]$. The set theoretic uncertainty is now given by

$$E_z = \left(\frac{0.5 - \lambda}{E_x} + \frac{0.5 + \lambda}{E_y} \right)^{-1} \quad . \tag{11}$$

However, x, y cannot be observed directly, but are corrupted by Gaussian noise. Hence, $\hat{z}$ is a random variable with statistics that can be obtained from (9). Since E_z in (11) does not depend on the actual observation, it is not a random variable. We provide three solutions:

- The exact density of $\hat{z}$ (Sec. 3),
- an approximation of the density by a sum (Sec. 4),
- and an exact second–order description, i.e., mean and variance (Sec. 5).

The first solution is mainly useful for validation purposes, the other two solutions are useful for data–recursive estimation.

3. EXACT SOLUTION FOR THE DENSITY

For $w_x \neq 0$, the density $f_z(z)$ of $\hat{z}$ is given by

$$f_z(z) = \frac{1}{w_x} \int_{-\infty}^{\infty} f_{xy}\left(\frac{z - w_y y}{w_x}, y \right) dy \quad . \tag{12}$$

From (1) we deduce the constraint

$$|x - y| \leq B \tag{13}$$

with

$$B = \sqrt{E_x} + \sqrt{E_y} \quad , \tag{14}$$

which leads to

$$f_{xy}(x, y) = \begin{cases} f_x(x)\, f_y(y) & \text{for} \quad |x - y| \leq B \\ 0 & \text{elsewhere} \end{cases} \quad . \tag{15}$$

The constraint can be rewritten as

$$-B \leq \frac{z - w_y y}{w_x} \leq B \quad \Longleftrightarrow \quad \frac{z - w_x B}{w_x + w_y} \leq y \leq \frac{z + w_x B}{w_x + w_y} \quad . \tag{16}$$

Hence, (12) gives

$$f_z(z) = \frac{1}{w_x} \int_{\frac{z - w_x B}{w_x + w_y}}^{\frac{z + w_x B}{w_x + w_y}} f_x\left(\frac{z - w_y y}{w_x} \right) f_y(y)\, dy \tag{17}$$

and after some manipulations

$$f_z(z) = \frac{1}{w_x} \frac{1}{2\pi \sigma_x \sigma_y} \exp\left\{ -\frac{1}{2} \frac{\left(z - (w_x m_x + w_y m_y)\right)^2}{w_x^2 S_x + w_y^2 S_y} \right\} I(z) \tag{18}$$

with

$$
\begin{aligned}
I(z) &= \int\limits_{\frac{z-w_x B}{w_x+w_y}}^{\frac{z+w_x B}{w_x+w_y}} \exp\left\{ \frac{w_x^2 S_x + w_y^2 S_y}{w_x^2 S_x S_y} \left(y - \frac{w_x^2 m_y S_x - w_x w_y m_x S_y + w_y S_y z}{w_x^2 S_x + w_y^2 S_y} \right)^2 \right\} dy \\
&= \frac{\sqrt{2\pi} w_x \sigma_x \sigma_y}{\sqrt{w_x^2 S_x + w_y^2 S_y}} \\
&\quad \left\{ \mathrm{erf}\left(\frac{\mathrm{Num}_1}{\sigma_x \sigma_y (w_x + w_y) \sqrt{w_x^2 S_x + w_y^2 S_y}} \right) \right. \\
&\quad \left. + \,\mathrm{erf}\left(\frac{\mathrm{Num}_2}{\sigma_x \sigma_y (w_x + w_y) \sqrt{w_x^2 S_x + w_y^2 S_y}} \right) \right\} \ ,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\mathrm{Num}_1 =& B(w_x^2 S_x + w_y^2 S_y) + z(w_x S_x - w_y S_y) \\
&+ w_x w_y (m_x S_y - m_y S_x) + w_y^2 m_x S_y - w_x^2 m_y S_x \ , \\
\mathrm{Num}_2 =& B(w_x^2 S_x + w_y^2 S_y) + z(w_y S_y - w_x S_x) \\
&+ w_x w_y (m_y S_x - m_x S_y) + w_x^2 m_y S_x - w_y^2 m_x S_y \ .
\end{aligned}
$$

The function $\mathrm{erf}\,(x)$ is not defined consistently in the literature. Here, it is defined according to[7]

$$
\mathrm{erf}\,(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{y=0}^{x} \exp\left(-\frac{y^2}{2} \right) dy \ . \tag{20}
$$

The exact density (18) can easily be calculated, but is not useful for recursive applications, because the derivation has been performed for the case that the primary stochastic uncertainties are Gaussian. Hence, the first approximation of the exact density $f_z(z)$ proposed in this paper is given by a sum of Gaussian densities.

4. APPROXIMATE SOLUTION FOR THE DENSITY

The key idea to finding an approximate solution for the probability density function is to look at the constraint (13) as $\mathrm{rect}\left(\frac{x-y}{B}\right)$ and to approximate this function by a weighted sum of Gaussian densities

$$
\mathrm{rect}\left(\frac{x-y}{B} \right) \approx \sum_{i=-L}^{L} \frac{1}{\sqrt{2\pi} c} \exp\left\{ -\frac{1}{2} \left(\frac{x - y - m_{g,i}}{\sigma_g} \right)^2 \right\}
$$

with $m_{g,i} = i\frac{B}{L}$, $\sigma_g = c\frac{B}{L}$, so that the integral from $-\infty$ to ∞ over the sum gives $2B$. The free parameter $c \in (0, \infty)$ may be obtained by a one–dimensional search to yield the best function approximation according to a given norm. A lot of manipulation reveals

$$
f_z\,(z) \approx \frac{\sum\limits_{i=-L}^{L} \left[\exp\left\{ -\frac{1}{2} \frac{(m_x - m_y - m_{g,i})^2}{S_g + S_x + S_y} \right\} \exp\left\{ -\frac{1}{2} \left(\frac{z - \bar{z}_i}{\eta} \right)^2 \right\} \right]}{\sum\limits_{i=-L}^{L} \sqrt{2\pi} \eta \exp\left\{ -\frac{1}{2} \frac{(m_x - m_y - m_{g,i})^2}{S_g + S_x + S_y} \right\}} \tag{21}
$$

with the individual means

$$
\begin{aligned}
\bar{z}_i \quad = \quad & \left(S_g \left[w_x m_x + w_y m_y\right]\right. \\
& + S_x \left[w_x \left(\quad m_{g,i} + m_y\right) + w_y m_y\right] \\
& + S_y \left[w_y \left(-m_{g,i} + m_x\right) + w_x m_x\right]\big) \\
& \Big/ \left(S_g + S_x + S_y\right) \quad ,
\end{aligned}
\tag{22}
$$

and equal variance

$$
\eta^2 = \frac{S_g \left(w_x^2 S_x + w_y^2 S_y\right) + \left(w_x + w_y\right)^2 S_x S_y}{S_g + S_x + S_y} \quad , \tag{23}
$$

where the shorthand notation $S_g = \sigma_g^2$ has been used. Using this approximate probability density function, we can easily calculate the approximate mean and variance of z. Furthermore, the *exact* mean and variance can be derived for $L \to \infty$.

5. EXACT SECOND–ORDER DESCRIPTION OF THE DENSITY

Given (21), the approximate mean is calculated as

$$
m_z \approx \frac{\sum\limits_{i=-L}^{L} \left[\exp\left\{-\frac{1}{2}\frac{(m_x - m_y - m_{g,i})^2}{S_g + S_x + S_y}\right\} \bar{z}_i\right]}{\sum\limits_{i=-L}^{L} \exp\left\{-\frac{1}{2}\frac{(m_x - m_y - m_{g,i})^2}{S_g + S_x + S_y}\right\}} \quad . \tag{24}
$$

The approximation of the variance is given by $\sigma_z^2 = \mathrm{E}\left\{z^2\right\} - m_z^2$ with

$$
\mathrm{E}\left\{z^2\right\} \approx \frac{\sum\limits_{i=-L}^{L} \left[\exp\left\{-\frac{1}{2}\frac{(m_x - m_y - m_{g,i})^2}{S_g + S_x + S_y}\right\} \left(\eta^2 + \bar{z}_i^2\right)\right]}{\sum\limits_{i=-L}^{L} \exp\left\{-\frac{1}{2}\frac{(m_x - m_y - m_{g,i})^2}{S_g + S_x + S_y}\right\}} \quad . \tag{25}
$$

Analytic expressions for the *exact* mean and variance are calculated from (24), (25) for $L \to \infty$, which implies $S_g \to 0$. The mean is then given by

$$
\begin{aligned}
m_z \quad = \quad & \int\limits_{v=-B}^{B} \exp\left\{-\frac{1}{2}\frac{\left(m_x - m_y - v\right)^2}{S_x + S_y}\right\} \frac{1}{S_x + S_y} \\
& \Big(S_x \left[w_x \left(v + m_y\right) + w_y m_y\right] + S_y \left[w_y \left(-v + m_x\right) + w_x m_x\right]\Big)\, dv \\
& \Bigg/ \int\limits_{v=-B}^{B} \exp\left\{-\frac{1}{2}\frac{\left(m_x - m_y - v\right)^2}{S_x + S_y}\right\} dv \\
= \quad & w_x m_x + w_y m_y - \left(w_x S_x - w_y S_y\right) \frac{\exp\left\{-\frac{1}{2}\frac{(B - m_x + m_y)^2}{S_x + S_y}\right\} - \exp\left\{-\frac{1}{2}\frac{(B + m_x - m_y)^2}{S_x + S_y}\right\}}{\sqrt{2\pi}\sqrt{S_x + S_y}\left(\mathrm{erf}\left(\frac{B - m_x + m_y}{\sqrt{S_x + S_y}}\right) + \mathrm{erf}\left(\frac{B + m_x - m_y}{\sqrt{S_x + S_y}}\right)\right)} \quad .
\end{aligned}
\tag{26}
$$

The variance is given by

$$
\begin{aligned}
S_z &= \int\limits_{v=-B}^{B} \exp\left\{-\frac{1}{2}\frac{(m_x - m_y - v)^2}{S_x + S_y}\right\} \frac{1}{(S_x + S_y)^2} \\
&\quad \left[\frac{(w_x + w_y)^2 S_x S_y}{S_x + S_y} + \left(S_x\left[w_x(v + m_y) + w_y m_y\right] + S_y\left[w_y(-v + m_x) + w_x m_x\right]\right)^2\right] dv \\
&\quad \Bigg/ \int\limits_{v=-B}^{B} \exp\left\{-\frac{1}{2}\frac{(m_x - m_y - v)^2}{S_x + S_y}\right\} dv - m_z^2 \\
&= w_x^2 S_x + w_y^2 S_y - (w_x S_x - w_y S_y)^2 Q_z \ , \qquad (27)
\end{aligned}
$$

where Q_z is defined according to

$$
Q_z = \frac{1}{2\pi(S_x + S_y)} \frac{\left(\exp\left\{-\frac{1}{2}\frac{(B - m_x + m_y)^2}{S_x + S_y}\right\} - \exp\left\{-\frac{1}{2}\frac{(B + m_x - m_y)^2}{S_x + S_y}\right\}\right)^2}{\left(\operatorname{erf}\left(\frac{B - m_x + m_y}{\sqrt{S_x + S_y}}\right) + \operatorname{erf}\left(\frac{B + m_x - m_y}{\sqrt{S_x + S_y}}\right)\right)^2} \qquad (28)
$$

$$
+ \frac{1}{\sqrt{2\pi}(S_x + S_y)^{3/2}} \frac{(B - m_x + m_y)\exp\left\{-\frac{1}{2}\frac{(B - m_x + m_y)^2}{S_x + S_y}\right\} + (B + m_x - m_y)\exp\left\{-\frac{1}{2}\frac{(B + m_x - m_y)^2}{S_x + S_y}\right\}}{\operatorname{erf}\left(\frac{B - m_x + m_y}{\sqrt{S_x + S_y}}\right) + \operatorname{erf}\left(\frac{B + m_x - m_y}{\sqrt{S_x + S_y}}\right)} \ .
$$

6. THE ASSEMBLED NEW ESTIMATOR

Given two uncertain information sources according to (1), it has been shown in Sec. 4, that the fusion result is given as the sum of a bounded uncertainty and a sum of Gaussian densities. When the number of terms included in the Gaussian sum tends towards infinity, the exact density derived in Sec. 3 is approached. This important result can now be applied to derive two different estimators for solving practical estimation problems.

The first estimator is obtained by keeping a finite number of, say M, terms in the Gaussian sum. Of course, when using this estimator recursively, there will be M^2 terms after the first recursion step. Hence, for recursive application, the number of terms must be kept fixed by selecting the M most important terms after each step.

An even simpler estimator is obtained, when using the additional results derived in Sec. 5, i.e., the exact second–order description of the density of the interval midpoint. The estimator is then given by the interval

$$
\mathcal{Z} = \{z : (z - \hat{z})^2 \leq E_z\} \ ,
$$

with a midpoint $\hat{z}$, that is uncertain in a stochastic sense. Mean and variance of $\hat{z}$ are given by

$$
m_z = w_x m_x + w_y m_y - (w_x S_x - w_y S_y) \frac{\exp\left\{-\frac{1}{2}\frac{(B - m_x + m_y)^2}{S_x + S_y}\right\} - \exp\left\{-\frac{1}{2}\frac{(B + m_x - m_y)^2}{S_x + S_y}\right\}}{\sqrt{2\pi}\sqrt{S_x + S_y}\left(\operatorname{erf}\left(\frac{B - m_x + m_y}{\sqrt{S_x + S_y}}\right) + \operatorname{erf}\left(\frac{B + m_x - m_y}{\sqrt{S_x + S_y}}\right)\right)} \qquad (29)
$$

from (26), and

$$
S_z = w_x^2 S_x + w_y^2 S_y - (w_x S_x - w_y S_y)^2 Q_z \qquad (30)
$$

from (27) with Q_z from (28). The set theoretic uncertainty width E_z is not a random variable and given by

$$
E_z = \left(\frac{0.5 - \lambda}{E_x} + \frac{0.5 + \lambda}{E_y}\right)^{-1} \qquad (31)
$$

from (11) for $\lambda \in [-0.5, 0.5]$

6.1. Selection of the Parameter λ

The parameter λ is selected in such a way, that an appropriate uncertainty measure is minimized. A good choice is, for example

$$\sqrt{E_z} + 3\sqrt{S_z} \ , \tag{32}$$

which is minimized numerically. There are other measures, where closed–form solutions for the minimizing λ can be found.

6.2. Border Cases of the New Estimator

The new estimator contains the classical estimation concepts, i.e., the set theoretic estimator and the Kalman filter, as border cases. We first investigate the case of a vanishing set theoretic uncertainty.

6.2.1. Border Case: Stochastic Uncertainty Only

This case is equivalent to $B = 0$. Hence, we have to examine the expression for m_z from (29) and S_z from (30) for the limit $B \to 0$. By applying the rule of l'Hospital, we obtain for the mean

$$m_z(B \to 0) = (w_x + w_y) \frac{S_x m_y + S_y m_x}{S_x + S_y} \ , \tag{33}$$

which can be simplified using $w_x + w_y = 1$ to yield

$$m_z(B \to 0) = \frac{\frac{m_y}{S_y} + \frac{m_x}{S_x}}{\frac{1}{S_x} + \frac{1}{S_y}} \ , \tag{34}$$

which is exactly the mean of a Kalman filter estimate. When examining (30), we obtain

$$S_z(B \to 0) = (w_x + w_y)^2 \frac{S_x S_y}{S_x + S_y} \ . \tag{35}$$

This is simplified by using $w_x + w_y = 1$ again to yield

$$S_z(B \to 0) = \left(\frac{1}{S_x} + \frac{1}{S_y} \right)^{-1} \ , \tag{36}$$

which is the variance of a Kalman filter estimate.

6.2.2. Border Case: Set Theoretic Uncertainty Only

The second border case is equivalent to $S_x = 0$, $S_y = 0$. Now we have to examine the expression for m_z from (29) and S_z from (30) for the limit $S_x \to 0$, $S_y \to 0$. For the mean, we obtain

$$m_z(S_x \to 0, S_y \to 0) = w_x m_x + w_y m_y \ . \tag{37}$$

The limiting variance of the new estimator vanishes, i.e.,

$$S_z(S_x \to 0, S_y \to 0) = 0 \tag{38}$$

as expected. Hence, for the border case of vanishing stochastic uncertainties, the new estimator converges to a set theoretic estimator according to (9), (10), (11).

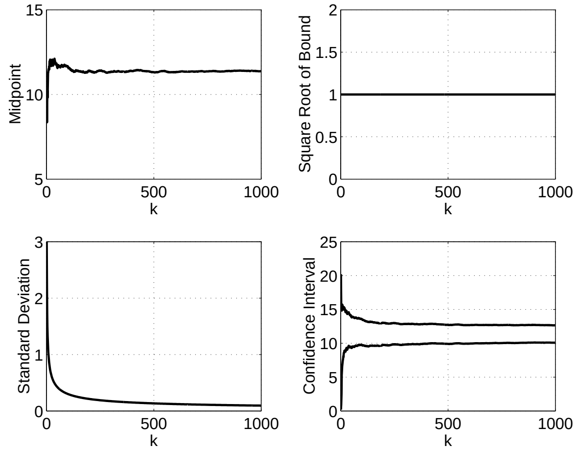

Figure 1. Simulative results for sampling the same source.

7. SIMULATIVE EXAMPLES

Consider estimation of x using measurements

$$y_k = x + e_k + c_k, \quad k = 1, \ldots, n \ , \tag{39}$$

where e_k is known to be bounded by $e_k^2 \leq E_k$ and $c_k \sim \mathrm{N}(0, \sigma_k)$. The recursive estimation scheme is initialized to

$$m(1) = y_1 \ , \quad E(1) = E_1 \ , \quad S(1) = \sigma_1^2 \ . \tag{40}$$

This estimate is recursively updated using (29), (30), and (31) with

$$m_x = m(k-1) \ , \quad E_x = E(k-1) \ , \quad S_x = S(k-1) \tag{41}$$

and

$$m_y = y_k \ , \quad E_y = E_k \ , \quad S_y = \sigma_k^2 \ . \tag{42}$$

Thus, the estimate

$$m(k) = m_z \ , \quad E(k) = E_z \ , \quad S(k) = S_z \tag{43}$$

incorporates the measurements $y_1, \ldots, y_k$. The parameter λ is selected according to Sec. 6.1.

7.1. Sampling the Same Source

For this simulation, the parameters $E_k = E = 1$, $\sigma_k = \sigma = 3$ (known to the filter) and $x = 12$, $e_k = e = -0.5$ (unknown to the filter) were used. Fig. 1 shows the resulting estimates $m(k)$, $\sqrt{E(k)}$, $\sigma(k) = \sqrt{S(k)}$, and the confidence interval $\left[m(k) - \sqrt{E(k)} - 3\sqrt{S(k)}, m(k) + \sqrt{E(k)} + 3\sqrt{S(k)}\right]$ for $k = 1, \ldots, 1000$. For an infinite number of measurements, this interval converges to $\left[x + e - \sqrt{E}, x + e + \sqrt{E}\right] = [10.5, 12.5]$, which is obviously correct. With the additional a priori knowledge that the error $e_k = e$ is constant, filtering could alternatively be divided into two subtasks: First, mean and variance of $x + e$ are calculated using pure stochastic filtering. Second, the bound of e allows for specification of the uncertainty of x. This approach yields the same interval for an infinite number of measurements.

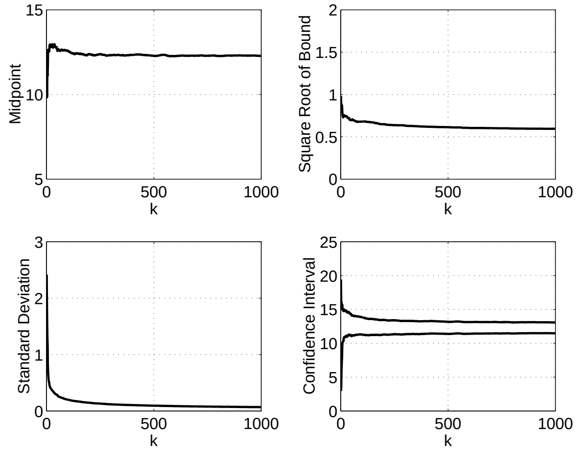

Figure 2. Simulative results for sampling different sources.

7.2. Sampling different sources

Here, we simulate an infinite number of information sources with set theoretic uncertainty $e_k = \sqrt{E_k}\, R_1(k)$, where $E_k = 0.25 + 0.75\, R_2(k)$, and stochastic noise $c_k \sim \mathrm{N}(0, \sigma_k)$ with $\sigma_k = \sqrt{1 + 8\, R_3(k)}$. $R_i(k)$, $i = 1, 2, 3$, are white random processes equally distributed in $[0, 1]$. Only the observation y_k, the error bound E_k of e_k, and the density of c_k are known to the filter. With additional a priori knowledge, i.e., the density of e_k, standard stochastic filtering would be applicable. But given only the bound E_k, this is not possible. The exact solution to this kind of filtering problems has been reported previously,[4,5] whereas (29), (30), and (31) yield an approximative solution. Fig. 2 shows the resulting estimates $m(k)$, $\sqrt{E(k)}$, $\sigma(k) = \sqrt{S(k)}$ and the confidence interval for $k = 1, \ldots, 1000$.

8. CONCLUSIONS

A vast class of estimation problems can be attacked as a mixed noise problem, i.e., the arising uncertainties can be modeled as being additively composed of both 1) noise with known bounds and 2) noise with known statistics. For these problems class, a new estimator has been derived, which combines set theoretic and stochastic estimation in a rigorous manner. Hence, it provides solution sets that are uncertain in a statistical sense. The proposed estimator is efficient and, hence, well–suited for practical applications. Our research also includes the vector case. However, this presentation has been restricted to the scalar case for ease of explanation.

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